T.5.5 [illegible]

Successive Differentiation

⇒ nth derivative of standard function

1. Find the nth derivatives of x^m, m is a +ve integer.

Soln: Let $y = x^m$

$$y_1 = \frac{dy}{dx} = m x^{m-1}$$

$$y_2 = m(m-1) x^{m-2}$$

$$y_3 = m(m-1)(m-2) x^{m-3}$$

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. . . .

$$y_n = m(m-1)(m-2)\ldots(m-n+1) x^{m-n}$$

$$\therefore y_n = \frac{m(m-1)(m-2)\ldots\ldots(m-n+1)(m-n)(m-n-1)\ldots 3.2.1}{(m-n)(m-n-1)\ldots\ldots 3.2.1} x^{m-n}$$

$$y_n = \frac{m!}{(m-n)!} x^{m-n}, \quad \text{where } n \le m$$

Note: ① If $n = m$ and $y = x^m$. Then $y_n = n!$ where m is a +ve integer.

② If $n > m$ and $y = x^m$ Then $y_n = 0$ where m is a +ve integer.

(2) Find the nth derivative's of $(ax+b)^m$ where m is a +ve Integer and $n \le m$.

Soln: Let $y = (ax+b)^m$

$y_1 = m(ax+b)^{m-1} \cdot a$

$y_2 = m(m-1)(ax+b)^{m-2} a^2$

$y_3 = m(m-1)(m-2)(ax+b)^{m-3} a^3$

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. . . .

$y_n = m(m-1)(m-2) \ldots (m-n+1)(ax+b)^{m-n} a^n$

$$\therefore y_n = \frac{m(m-1)(m-2)\ldots(m-n+1)(m-n)(m-n-1)\ldots 3\cdot 2\cdot 1}{(m-n)(m-n-1)\ldots 3\cdot 2\cdot 1} (ax+b)^{m-n} \cdot a^n$$

$$\therefore y_n = \frac{m!}{(m-n)!} (ax+b)^{m-n} \cdot a^n.$$

(3) H.W Find the nth derivative of $\frac{1}{(ax+b)^m}$ where m is a +ve Integer ~~and~~.

Let $y = \frac{1}{(ax+b)^m} \Rightarrow y = (ax+b)^{-m}$

(3) find The nth derivative of $\frac{1}{ax+b}$

Soln: Let $y = \frac{1}{(ax+b)}$

$y = (ax+b)^{-1}$

$y_1 = (-1)(ax+b)^{-2} \cdot a \rightarrow (1)$

$y_2 = (-1)(-2)(ax+b)^{-3} \cdot a^2$

$\Rightarrow (-1)^2 \cdot 2!\ (ax+b)^{-3} \cdot a^2 \rightarrow (2)$

$y_3 = (-1)(-2)(-3)(ax+b)^{-4} \cdot a^3$

$\Rightarrow (-1)^3 \cdot 3!\ (ax+b)^{-4} \cdot a^3$

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. . . .

$y_n = (-1)^n \cdot n!\ (ax+b)^{-(n+1)} \cdot a^n.$

$$y_n = \frac{(-1)^n n!\ a^n}{(ax+b)^{(n+1)}}$$

(4) H.W: find The nth derivative of $\frac{1}{(ax+b)^m}$ where 'm' is a +ve integer.

Soln: Let $y = \frac{1}{(ax+b)^m}$

$y = (ax+b)^{-m}.$

(a) Find the nth derivative of $\frac{1}{(ax+b)^m}$ where m is a +ve Integer.

Sol^n: Let $y = \frac{1}{(ax+b)^m}$

$$y = (ax+b)^{-m}$$

$$y_1 = (-m)\,(ax+b)^{-m-1} \cdot a$$

$$y_2 = (-m)(-m-1)\,(ax+b)^{-m-2} \cdot a^2$$

$$y_2 = (-m)(-1)(m+1)\,(ax+b)^{-m-2} \cdot a^2$$

$$y_2 = (-1)^2 (m)(m+1)\,(ax+b)^{-m-2} \cdot a^2$$

$$y_3 = (-m)(-m-1)(-m-2)\,(ax+b)^{-m-3} \cdot a^3$$

$$y_3 = (-1)^3 \cdot m(m+1)(m+2)\,(ax+b)^{-m-3} \cdot a^3$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$y_n = (-1)^n m(m+1)(m+2)\ldots(m+n-1)\,(ax+b)^{-m-n} \cdot a^n$$

$$\therefore y_n = \frac{(-1)^n\, 1.2.3.\ldots(m-1)(m)(m+1)(m+2)\ldots(m+n-1)}{1.2.3\ldots\ldots(m-1)\;(ax+b)^{m-n}} \cdot a^n.$$

$$y_n = \frac{(-1)^n (m+n-1)!}{(m-1)!\,(ax+b)^{m+n}}$$

(5) Find the nth derivative of $\log(ax+b)$

Sol^n: Let $y = \log(ax+b)$

$$y_1 = \frac{1}{(ax+b)} \cdot a$$

$$y_1 = (ax+b)^{-1} \cdot a$$

$$y_2 = (-1)(ax+b)^{-2} \cdot a^2$$

$$y_3 = (-1)(-2)(ax+b)^{-3} \cdot a^3$$

$$\Rightarrow (-1)^2\, 2!\, (ax+b)^{-3}\, a^3$$

$$y_4 = (-1)(-2)(-3)(ax+b)^{-4} \cdot a^4$$

$$\Rightarrow (-1)^3\, 3!\, (ax+b)^{-4} \cdot a^4$$

$$y_5 = (-1)(-2)(-3)(-4)(ax+b)^{-5}\, a^5$$

$$\Rightarrow (-1)^4 \cdot 4!\, (ax+b)^{-5}\, a^5$$

. . . .

. . . .

$$y_n = (-1)^{n-1} \cdot (n-1)!\, (ax+b)^{-n}\, a^n$$

$$y_n = \frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$$

$\therefore \boxed{x = 1070}$

⑥ Find the n^{th} derivative of e^{ax}.

Soln! Let $y = e^{ax}$

$y_1 = a \cdot e^{ax}$

$y_2 = a^2 e^{ax}$

$y_3 = a^3 \cdot e^{ax}$

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$y_n = a^n (e^{ax})$

$\Rightarrow \dfrac{d^n(e^{ax})}{dx^n} = a^n e^{ax}$

⑦ Find The n^{th} derivative of $\sin(ax+b)$

Soln: $y = \sin(ax+b)$

$y_1 = a \cos(ax+b)$

$y_1 = a \sin(ax+b+\pi/2)$

$y_2 = a^2 \cos(ax+b+\pi/2)$

$\Rightarrow a^2 \sin(ax+b+\pi/2+\pi/2)$

$y_2 = a^2 \sin(ax+b+2 \cdot \pi/2)$

. . . .

. . . .

$y_n = a^n \sin(ax+b+n \cdot \pi/2)$

* Find the nth derivative of $\cos(ax+b)$

Soln: let $y = \cos(ax+b)$

$y' = -\sin(ax+b) \cdot a$

{Note :- If $-\sin x = \cos(\frac{\pi}{2} + x)$ }

$\therefore y' = a \cdot \cos(\frac{\pi}{2} + ax + b)$

$y'' = a^2 (-\sin(\frac{\pi}{2} + ax + b))]$

$y'' = a^2 \cos(\frac{\pi}{2} + \frac{\pi}{2} + ax + b)]$

$y'' = a^2 \cos(2\frac{\pi}{2} + ax + b)$

$\vdots$

$y^n = a^n \cdot \cos(\frac{n\pi}{2} + ax + b)$ //

* Find the nth derivative of $e^{ax}\sin(bx+c)$

Soln: let $y = e^{ax} \cdot \sin(bx+c)$

$y' = e^{ax} \cdot \cos(bx+c) \cdot b + \sin(bx+c) \cdot a e^{ax}$

$y' = e^{ax}[b \cdot \cos(bx+c) + a \sin(bx+c)]$

let $a = r\cos\theta$, $b = r\sin\theta$

$a^2 + b^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$\Rightarrow r^2[\cancel{\cos^2\theta + \sin^2\theta}^{\,1}]$

$a^2 + b^2 \Rightarrow r^2$

$\therefore \boxed{r = \sqrt{a^2+b^2}}$ and $\theta = \tan^{-1}(\frac{b}{a})$

$\therefore\ y' = e^{ax}[r\sin\theta.\ \cos(bx+c) + r\cos\theta.\ \sin(bx+c)]$

$y' = e^{ax}.\ r[\sin\theta.\ \cos(bx+c) + \cos\theta.\ \sin(bx+c)] \longrightarrow (1)$

$\Updownarrow$

{Note: $\sin(x+y) = \sin x.\ \cos y + \cos x.\ \sin y$} $\longrightarrow (2)$

Comparing Eq (1) & (2), Then

$y' = e^{ax}.\ r\ \sin[(bx+c)+\theta] \longrightarrow (3)$

Thus The result [Equation (3)] Implies that the derivative "y'" can be obtained in the form of the function "y" on multiplying with by "r" & increasing angle by "θ".

Repeating the same procedure.

$\therefore\ y_2 = e^{ax}.\ r^2 \sin[(bx+c)+2\theta]$

$y_3 = e^{ax}.\ r^3 \sin[(bx+c)+3\theta]$

$\vdots$

$\therefore\ y_n = e^{ax}.\ r^n \sin[(bx+c)+n\theta]$

(4) Find the nth derivative of $e^{ax} \cdot \cos(bx+c)$

Sol:- Let $y = e^{ax} \cdot \cos(bx+c)$

$$y' = e^{ax}[-\sin(bx+c) \cdot b] + \cos(bx+c) \cdot a e^{ax}.$$

$$y' = e^{ax}[a \cdot \cos(bx+c) - b \cdot \sin(bx+c)]$$

put $a = r\cos\theta$, $b = r\sin\theta$

$$a^2+b^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$r^2[\underbrace{\cos^2\theta + \sin^2\theta}_{1}]$$

$$a^2+b^2 = r^2$$
$$\therefore r = \sqrt{a^2+b^2} \quad \text{and}$$
$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore y' = e^{ax}[r\cos\theta \cdot \cos(bx+c) - r\sin\theta \cdot \sin(bx+c)]$$

$$y' = e^{ax} \cdot r[\cos\theta \cdot \cos(bx+c) - \sin\theta \cdot \sin(bx+c)] \rightarrow (1)$$

{Note:- $\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$} $\rightarrow (2)$

Comparing Equation (1) & (2) Then

$$y' = e^{ax} \cdot r \cos[(bx+c)+\theta] \longrightarrow (3)$$

The above Equation (3) result's implies that the derivative of "y'" can be obtained in the form of the function "y", on multiplying with by "r" and increasing angle by "θ".

$\therefore$ Repeating the Same procedure

$$y'' = e^{ax} \cdot r^2 \cdot \cos[(bx+c)+2\theta]$$

$$y''' = e^{ax} \cdot r^3 \cos[(bx+c)+3\theta]$$

$\vdots$

$$y^n = e^{ax} \cdot r^n \cos[(bx+c)+n\theta]$$

* Find the nth derivative of a^{mx}.

Soln: let $y = a^{mx}$

$$y' = m\, a^{mx} \log a.$$

$$y'' = m^2 a^{mx} (\log a)^2$$

$$y''' = m^3 a^{mx} (\log a)^3$$

$\vdots$

$$y^n = m^n \cdot a^{mx} (\log a)^n$$

① Find the nth derivative of $\log(x^2-9)$

Soln: let $y = \log(x^2-9)$

$y = \log(x^2-3^2)$

$y = \log[(x+3)(x-3)]$

$(a^2-b^2) = (a+b)(a-b)$

$\log(ab) = \log a + \log b$

$y = \log(x+3) + \log(x-3)$

Applying nth derivative on both side

$$y^n = \left[\frac{d^n}{dx^n}(\log(x+3)) + \frac{d^n}{dx^n}\log(x-3)\right]$$

$$\left\{ \text{Note: } \frac{d^n}{dx^n}\log(ax+b) = \frac{(-1)^{n-1}(n-1)!\cdot a^n}{(ax+b)^n} \right.$$

$$\therefore \frac{d^n}{dx^n}\log(x+3) = \frac{(-1)^{n-1}(n-1)!}{(x+3)^n}$$

$$\text{ılly } \frac{d^n}{dx^n}\log(x-3) = \frac{(-1)^{n-1}(n-1)!}{(x-3)^n}$$

$$y_n = \left[\frac{(-1)^{n-1}(n-1)!}{(x+3)^n} + \frac{(-1)^{n-1}(n-1)!}{(x-3)^n}\right]$$

$$y_n = (-1)^{n-1}(n-1)!\left[\frac{1}{(x+3)^n} + \frac{1}{(x-3)^n}\right]$$

(2) Find the nth derivative of $\log(x^2+ax)$

Soln: let $y = \log(x^2+ax)$

$$y = \log[x(x+a)]$$

$$y = \log x + \log(x+a)$$

Applying nth derivative on both side

$$y^n = \left[\frac{d^n}{dx^n}\log(x) + \frac{d^n}{dx^n}\log(x+a)\right]$$

$$\left.\begin{aligned}\frac{d^n}{dx^n}\log(x) &= \frac{(-1)^{n-1}(n-1)!}{x^n}\\ \frac{d^n}{dx^n}\log(x+a) &= \frac{(-1)^{n-1}(n-1)!}{(x+a)^n}\end{aligned}\right\}$$

$$y^n = \left[\frac{(-1)^{n-1}(n-1)!}{x^n} + \frac{(-1)^{n-1}(n-1)!}{(x+a)^n}\right]$$

$$y^n = (-1)^{n-1}(n-1)!\left[\frac{1}{x^n} + \frac{1}{(x+a)^n}\right]$$

① Find the nth derivative of $\log(5x-1)$

Soln: let $y = \log(5x-1)$

Applying nth derivative on Both side.

Then $y^n = \frac{d^n}{dx^n} \log(5x-1)$

$$y^n = \frac{(-1)^{n-1}(n-1)!\cdot 5^n}{(5x-1)^n}$$

$\log(ax+b) = \frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$

where $(ax+b) = (5x-1)$

$a=5$

② Find nth derivative of $\log(x^2-4)$

Soln: let $y = \log(x^2-4)$

$y = \log(x^2-2^2)$

$\boxed{y = \log(a+b)(a-b)}$

$y = \log[(x+2)(x-2)]$

$y = \log(x+2) + \log(x-2)$

Applying nth derivative on Both equation

$\therefore y^n = \frac{d^n}{dx^n}\log(x+2) + \frac{d^n}{dx^n}\log(x-2)$

$$y^n = \frac{(-1)^{(n-1)}(n-1)!}{(x+2)^n} + \frac{(-1)^{(n-1)}(n-1)!}{(x+2)^n}$$

$$y^n = (-1)^{(n-1)}(n-1)!\left[\frac{1}{(x+2)^n} + \frac{1}{(x-2)^n}\right]$$

(*) Find nth derivative of $\log\left(1+\frac{1}{x}\right)$

Sol^n : Let $y = \log\left[1 + 1/x\right]$

$y = \log\left[\frac{(x+1)}{x}\right]$

$y = \log(x+1) - \log x$

$\log\left(\frac{a}{b}\right) \Rightarrow \log a - \log b$

Apply nth derivative on Both Side Then

$$y^n = \left[\frac{d^n}{dx^n}\log(x+1) - \frac{d^n}{dx^n}\log(x)\right]$$

$$y^n = \left[\frac{(-1)^{n-1}(n-1)!}{(x+1)^n} - \frac{(-1)^{n-1}(n-1)!}{x^n}\right]$$

$$y^n = (-1)^{n-1}(n-1)!\left[\frac{1}{(x+1)^n} - \frac{1}{x^n}\right]$$