Equation :- In mathematics, An eqn is a statement that asserts the equality of two expressions, which are connected by the equal sign "=".

Ex :- ① $8 + 2 = 12 - 2$

② $Ax^2 + Bx + C = y$

③ $x + y = 2$

Differential Equation :- An eqn containing independent variable and dependent variable and the derivative of dependent variable w.r.t independent variable is called a D.E.

Ex :- ① $\frac{dy}{dx} = x e^x$

② $\frac{d^2y}{dx^2} = -a^2 x$

③ $y = x \frac{dy}{dx} + \frac{1}{\frac{dy}{dx}}$

④ $x \frac{dy}{dx} + y = 0$

⑤ $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$

Order of D.E :- The order of D.E is the order of the highest derivative present in the equation.

Ex :- 1) $x^2 \frac{d^3y}{dx^3} + 6x \frac{d^2y}{dx^2} + 10y = 0$ order = 3

2) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 = e^{4x}$ order = 3

3) $y''(x) + \lambda y(x) = 0$, $\lambda \in R$ order = 2

4) $x'(t) + x(t) = 0$. order = 1

Degree of D.E :- The Degree of D.E is the +ve integral power of the highest order derivative present in the eqn.

Ex :- 1) $\left(\frac{d^2y}{dx^2}\right)^3 + 5\left(\frac{dy}{dx}\right) = e^x$. Degree = 3

2) $\left(\frac{d^3y}{dx^3}\right)^2 + \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 = e^{4x}$ Degree = 2

Classification of D.E :-

There are 2 types of D.E. They are

1) Ordinary Differential Equation
2) Partial Differential Equation.

1) Ordinary Differential Equation :-

A Differential Equation involving total derivative of only one independent variable and one or more dependent variable is called an O.D.E.

Ex :- 1) $\frac{dy}{dx} + y = e^x$

2) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \cos x$

} Simple O.D.E

3) $\frac{dy_1}{dx} + \frac{dy_2}{dx} = e^x$

4) $\frac{dy_1}{dx} - \frac{dy_2}{dx} = e^{-x}$

} System of O.D.E.

2) Partial Differential Equation :-

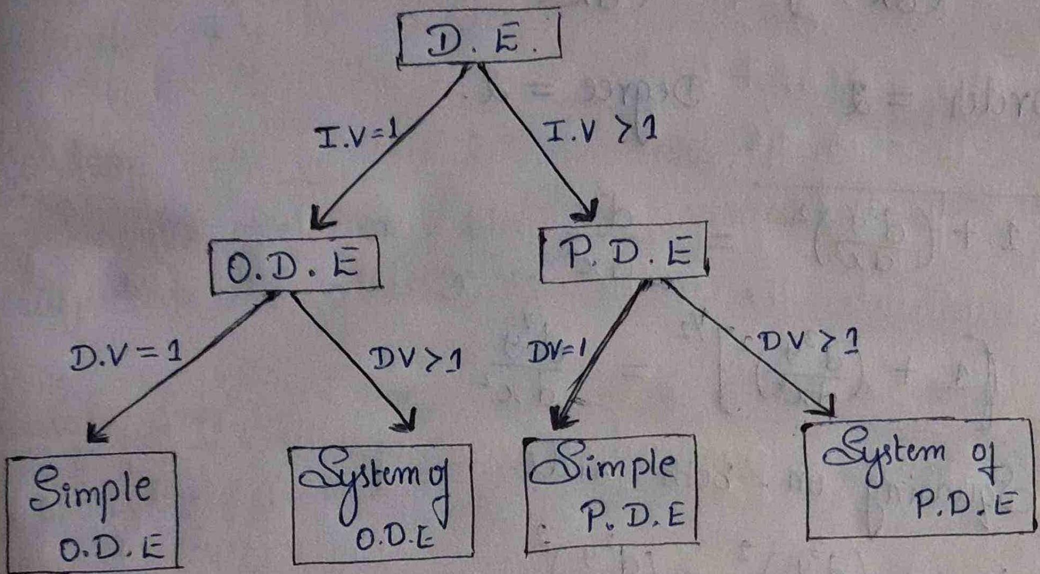
A Differential Equation involving partial derivative of more than one independent variable w.r.t one or more dependent variable is called a P.D.E

Ex :- 1) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 1 \longrightarrow$ Simple P.D.E

2) $\dfrac{\partial z_1}{\partial x} + \dfrac{\partial z_2}{\partial y} = x + y$

3) $\dfrac{\partial z_1}{\partial x} - \dfrac{\partial z_2}{\partial y} = x - y$

} System of P.D.E.



Find the order and Degree of the following D.E:-

1) $\left(\dfrac{d^2y}{dx^2}\right)^3 + 5\left(\dfrac{dy}{dx}\right) = e^x$

soln :- order = 2 Degree = 3.

2) $y = x\dfrac{dy}{dx} + \dfrac{1}{\left(\frac{dy}{dx}\right)}$

soln:- Multiply by $\left(\dfrac{dy}{dx}\right)$ on both sides

$y\left(\dfrac{dy}{dx}\right) = x\left(\dfrac{dy}{dx}\right)^2 + 1.$

order = 1. Degree = 2.

2m

3) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2}$

soln :- Squaring on both side

$$\left(\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/\cancel{2}}\right)^{\cancel{2}} = \left(\frac{d^2y}{dx^2}\right)^2$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

order = 2 , Degree = 2.

2m

4) $\sqrt{1+\left(\frac{d^3y}{dx^3}\right)^2} = \frac{d^2y}{dx^2}$.

soln :- $\left[1+\left(\frac{d^3y}{dx^3}\right)^2\right]^{1/2} = \frac{d^2y}{dx^2}$

Squaring on both side.

$$1+\left(\frac{d^3y}{dx^3}\right)^2 = \left(\frac{d^2y}{dx^2}\right)^2$$

order = 3 Degree = 2

5) $y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + 1$

soln :- order = 1 , Degre = 2.

6) $y = x\left(\frac{dy}{dx}\right) + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

soln : $y - x\frac{dy}{dx} = a\left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2}$

Squaring on both side

$$\left(y - x\frac{dy}{dx}\right)^2 = a^2\cdot\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

order = 1 Degree = 2.

7) $\left(1 + \frac{dy}{dx}\right)^4 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

soln: order = 2 Degree = 2

8) $\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0$

soln: order = 3 Degree = 1

9) $\left(\frac{d^4y}{dx^4}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 + y = \sin x$

soln: order = 4 Degree = 2

Solution of a D.E :- A soln of a D.E is a functional relation b/w the variables involved which satisfied the given D.E.

General Solution :- The soln of a D.E in which the no of arbitrary constant is equal to the order of the D.E.

Particular Solution :- If the Particular values are given to the arbitrary constants in the general soln, then the soln so obtained is called a Particular soln

Ex :- Consider $\frac{dy}{dx} = \frac{3}{2y}$

Apply variable Separable method, we get

$$y\,dy = \frac{3}{2}dx$$

on Integration

$$\frac{y^2}{2} = \frac{3}{2}x + C$$

$$y^2 = 3x + C$$

This is the soln of the D.E

In this soln only one arbitrary const present i.e., 'C'. and the order of the given D.E is 1

$\therefore$ order of the D.E = arbitrary const present in the soln

$\therefore$ $y^2 = 3x + C$ is the general soln of D.E.

To find C :-

put $y = 4$ and $x = 5$ in given soln, we get

$(4)^2 = 3(5) + C$

$16 = 15 + C$

$\boxed{C = 1}$

$\therefore$ Substitute the value of C in soln, we get

$y^2 = 3x + 1$

This is the particular soln of the D.E.

First order and First Degree Differential Equation :-

A D.E of first order and first degree can be written as $f(x, y, \frac{dy}{dx}) = 0$

Solution of first order and first degree D.E :-

It is not possible to solve the first order and first degree differential eqn in general. So we can use some special methods to solve such eqns. A first order and first degree differential eqn can be solved using the following Methods.

1) Equations where variables are separable.
2) Homogeneous equations.
3) Exact differential Equations.
4) Linear differential Equations.

Method 1 :- Variable Separable

If in an eqn it is possible to collect all funcs of x and dx on one side and all funcs of y and dy on other side, then the variable are said to be separable. Thus the general form of such an eqn is

$$f(x)\,dx = g(y)\,dy.$$

Integrating on both sides we get

$$\int f(x)\,dx = \int g(y)\,dy + c \quad \text{is solution.}$$

where c is an integrating const.

Problems:-

1) Solve $\dfrac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$

soln:- Given eqn is $\dfrac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$

$$\frac{dy}{dx} = e^{3x} \cdot e^{-2y} + x^2 e^{-2y}$$

$$\frac{dy}{dx} = e^{-2y}\left(e^{3x} + x^2\right)$$

Separate the variable, we get.

$$\frac{dy}{e^{-2y}} = \left(e^{3x} + x^2\right) dx$$

$$e^{2y}\,dy = e^{3x}\cdot dx + x^2\,dx \quad \text{on integration, we get}$$

$$\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + C$$

$$\frac{e^{2y}}{2} = \frac{e^{3x} + x^3 + 3C}{3}$$

$$3e^{2y} = 2e^{3x} + 2x^3 + 6C$$

$$3e^{2y} = 2(e^{3x} + x^3) + c_1 \quad \text{where } c_1 = 6C.$$

This is the soln of given D.E.

2) $\frac{dy}{dx} = xy + x + y + 1$

soln :- $\frac{dy}{dx} = x(y+1) + (y+1)$

$\frac{dy}{dx} = (x+1)(y+1)$

Separate the variable, we get

$\frac{dy}{y+1} = (x+1)\,dx$

on integration

$\log(y+1) = \frac{x^2}{2} + x + C$ is the soln of given D.E.

HW

3) $\frac{dy}{dx} = e^{x-y} + e^{2\log x - y}$

soln :- $\frac{dy}{dx} = e^x \cdot e^{-y} + e^{2\log x} \cdot e^{-y}$

$\left\{ \begin{array}{l} a\log b = \log b^a \\ e^{\log a} = a \end{array} \right\}$

$= e^{-y}\left[e^x + e^{\log x^2}\right]$

$\frac{dy}{dx} = e^{-y}\left[e^x + x^2\right]$

Separate the variable we get.

$\frac{dy}{e^{-y}} = \left[e^x + x^2\right] dx \Rightarrow e^y\,dy = e^x\,dx + x^2\,dx$

On Integration, we get

$e^y = e^x + \frac{x^3}{3} + C$ is the soln

2m

4) $\sec^2 x \cdot \tan y\,dx + \sec^2 y \cdot \tan x\,dy = 0$

soln :- Given D.E is $\sec^2 x \cdot \tan y\,dx + \sec^2 y \cdot \tan x\,dy = 0$

$\sec^2 x \cdot \tan y\,dx = -\sec^2 y \cdot \tan x\,dy$

Separate the variable, we get

$$\frac{\sec^2 x}{\tan x} \cdot dx = -\frac{\sec^2 y}{\tan y} \cdot dy$$

Integrating on both side, we get $\quad \int \frac{f'(x)}{f(x)} dx = f(x) + c$

$$\int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy$$

$$\therefore \log(\tan x) = -\log(\tan y) + \log c$$

$$\log(\tan x) + \log(\tan y) = \log c \quad [\log a + \log b = \log(ab)]$$

$$\log[(\tan x)\cdot(\tan y)] = \log c$$

$(\tan x)\cdot(\tan y) = c$ is the soln.

5m

5) Solve $(x^2 - yx^2)\frac{dy}{dx} + (y^2 + xy^2) = 0$

soln:-

$$(x^2 - yx^2)\frac{dy}{dx} = -(y^2 + xy^2)$$

$$x^2(1-y)\frac{dy}{dx} = -y^2(1+x)$$

$$\frac{(1-y)}{y^2} dy = -\frac{(1+x)}{x^2} dx$$

~~on~~ Integrating on both side, we get.

$$\int \frac{1}{y^2} dy - \int \frac{1}{y} dy = -\int \frac{1}{x^2} dx - \int \frac{1}{x} dx$$

$$-\frac{1}{y} - \log y = -\left(-\frac{1}{x}\right) - \log x + c$$

$$-\frac{1}{y} - \log y = \frac{1}{x} - \log x + c$$

$$-\frac{1}{y} - \frac{1}{x} - \log y + \log x = c \quad \left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$$

$-\left[\frac{1}{x} + \frac{1}{y}\right] + \log\left(\frac{x}{y}\right) = c$ is the soln.

HW

6) $\frac{dy}{dx} = \frac{xy+y}{xy+x}$

soln :- $\frac{dy}{dx} = \frac{y(x+1)}{x(y+1)}$

Separate the variable, we get

$$\frac{(y+1)}{y} dy = \frac{(x+1)}{x} dx$$

$$\left(1 + \frac{1}{y}\right) dy = \left(1 + \frac{1}{x}\right) dx$$

Integrating on both sides, we get.

$$y + \log y = x + \log x + c.$$

$$y - x + \log y - \log x = c$$

$$y - x + \log\left(\frac{y}{x}\right) = c$$ is the soln

HW

7) Solve $(1+x^2) dy + x\sqrt{1-y^2}\, dx = 0$

soln :- Given eqn is $(1+x^2) dy + x\sqrt{1-y^2}\, dx = 0.$

Separate the variable, we get

$$\cancel{(1+x^2)}\ (1+x^2)\, dy = -x\sqrt{1-y^2}\, dx$$

$$\frac{dy}{\sqrt{1-y^2}} = \frac{-x}{(1+x^2)} dx$$

$$\frac{dy}{\sqrt{1-y^2}} + \frac{2x}{2(1+x^2)} dx = 0$$

Here multiply & divided by 2.

Integrating on both sides, we get.

$$\int \frac{1}{\sqrt{1-y^2}} dy + \frac{1}{2}\int \frac{2x}{\cancel{2}(1+x^2)} dx = 0.$$

$$\sin^{-1} y + \frac{1}{2}\log(1+x^2) = C$$ is the soln.

8) Solve 3

soln :- Given

$3e^x$ tan

$\frac{3e^x}{(1-e^x}$

Integ

$-3\log$

log(tan

log

9) Solve

Soln :-

$y[$

$(y +$

8) Solve $3e^x \tan y \, dx + (1-e^x)\sec^2 y \, dy = 0$

soln :- Given D.E is $3e^x \tan y \, dx + (1-e^x)\sec^2 y \, dy = 0$

$$3e^x \tan y \, dx = -(1-e^x)\sec^2 y \, dy$$

$$\frac{3e^x}{(1-e^x)} \cdot dx = -\frac{\sec^2 y}{\tan y} \cdot dy$$

Integrating on both side, we get

$$-3\log(1-e^x) = -\log(\tan y) + \log c$$

$$\log(\tan y) - 3\log(1-e^x) = \log c$$

$$\log(\tan y) - \log(1-e^x)^3 = \log c$$

$$\log\left[\frac{\tan y}{(1-e^x)^3}\right] = \log c$$

$$\frac{\tan y}{(1-e^x)^3} = c$$

$\underline{\tan y = c(1-e^x)^3}$ is a soln.

9) Solve $x^{-1}\cos^2 y \, dy + y^{-1}\cos^2 x \, dx = 0$

Soln :- $x^{-1}\cos^2 y \, dy + y^{-1}\cos^2 x \, dx = 0$

$$\frac{\cos^2 y}{x} dy + \frac{\cos^2 x}{y} dx = 0$$

$$\frac{y\cos^2 y \, dy + x\cos^2 x \, dx}{xy} = 0$$

$$y\cos^2 y \, dy + x\cos^2 x \, dx = 0$$

$$y\left[\frac{1+\cos 2y}{2}\right] dy + x\left[\frac{1+\cos 2x}{2}\right] dx = 0.$$

$$(y + y\cos 2y)\, dy + (x + x\cos 2x)\, dx = 0$$

Integrating on both side, we get

$$\int y\,dy + \int \underset{u}{y}\,\underset{v}{\cos 2y}\,dy + \int x\,dx + \int x\cos 2x\,dx = 0$$

$$\left\{ \int uv = u\int v - \int\int v\,.\,du \right\}$$

$$\frac{y^2}{2} + y\int \cos 2y\,dy - \int\int \cos 2y\,dy\,(1) + \frac{x^2}{2} + x\int x\cos 2x - \int\int \cos 2x\,.\,dx = C$$

$$\frac{y^2}{2} + y\left(\frac{\sin 2y}{2}\right) - \int \frac{\sin 2y}{2}\,.\,dy + \frac{x^2}{2} + x\left(\frac{\sin 2x}{2}\right) - \int \frac{\sin 2x}{2}\,dx = C$$

$$\frac{y^2}{2} + \frac{y\sin 2y}{2} - \frac{1}{2}\left(\frac{-\cos 2y}{2}\right) + \frac{x^2}{2} + \frac{x\sin 2x}{2} - \frac{1}{2}\left(\frac{-\cos 2x}{2}\right) = C$$

$$\frac{2y^2 + 2y\sin 2y + \cos 2y + 2x^2 + 2x\sin 2x + \cos 2x}{4} = C$$

$$2y^2 + 2y\sin 2y + \cos 2y + 2x^2 + 2x\sin 2x + \cos 2x = 4C$$

This is the soln of given D.E.

10) $\frac{dy}{dx} + xy = xy^3$

soln :- $\frac{dy}{dx} = xy^3 - xy$

$$\frac{dy}{dx} = x(y^3 - y)$$

$$\frac{dy}{y^3 - y} = x\,dx$$

$$\frac{dy}{y(y^2-1)} = x\,dx$$

Integrating on both side, we get.

$$\int \frac{1}{y(y^2-1)}\,dy = \int x\,dx$$

~~Multi~~ add and substract y^2 in LHS we get

$$\int \frac{1 - y^2 + y^2}{y(y^2-1)}\,dy = \frac{x^2}{2} + C$$

$$\int \frac{-(y^2-1) + y^2}{y(y^2-1)}\,dy = \frac{x^2}{2} + C$$

$$-\int \frac{\cancel{(y^2-1)}}{y\cancel{(y^2-1)}}\,dy + \int \frac{y^{\cancel{2}}}{\cancel{y}(y^2-1)}\,dy = \frac{x^2}{2} + C$$

$$-\int \frac{1}{y}\,dy + \int \frac{y}{y^2-1}\,dy = \frac{x^2}{2} + C$$

Here multiple and $\div$ by 2.

$$-\log y + \frac{1}{2}\int \frac{2y}{y^2-1}\,dy = \frac{x^2}{2} + C$$

$$-\log y + \frac{1}{2}\log(y^2-1) = \frac{x^2}{2} + C$$

$$-\log y + \log\sqrt{y^2-1} = \frac{x^2}{2} + C$$

$$\log\left[\frac{\sqrt{y^2-1}}{y}\right] = \frac{x^2}{2} + C$$

OR

$$-\log y + \frac{1}{2}\log(y^2-1) = \frac{x^2}{2} + C$$

$$\frac{-2\log y + \log(y^2-1)}{2} = \frac{x^2 + 2C}{2}$$

$$-\log y^2 + \log(y^2-1) = x^2 + C_1 \quad \text{where } C_1 = 2C$$

$$\log\left(\frac{y^2-1}{y^2}\right) = x^2 + C_1 \text{ is the soln.}$$

11) $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}\,dy = 0$,

soln :- $e^x\sqrt{1-y^2}\,dx = -\frac{y}{x}\,dy$

$$x e^x\,dx = -\frac{y}{\sqrt{1-y^2}}\,dy$$

$$x e^x\,dx + \frac{y}{\sqrt{1-y^2}}\,dy = 0.$$

Integrating on both side, we get

$$\int x e^x\,dx + \int \frac{y}{\sqrt{1-y^2}}\,dy = C$$

put $1-y^2 = t$

Diff w.r.t 'x' on b.s.

$$-2y\,dy = dt$$

$$dy = \frac{-1}{2y}\,dt$$

$$\int \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{e^x}\,dx + \int \frac{\cancel{y}}{\sqrt{t}}\left(-\frac{1}{2\cancel{y}}\,dt\right) = C$$

$$x e^x - \int e^x (1)\,dx - \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt = C$$

$$x e^x - e^x - \frac{1}{\cancel{2}}(\cancel{2}\sqrt{t}) = C$$

$$e^x(x-1) - \sqrt{t} = C$$

$e^x(x-1) - \sqrt{1-y^2} = C$ is the soln

12) Solve $y\,dx + (1+x^2)\tan^{-1}x\,dy = 0$

soln:- $y\,dx = -(1-x^2)\tan^{-1}x\,dy$

$$\frac{dx}{(1+x^2)\tan^{-1}x} = \frac{-dy}{y}$$

Integrating on both side, we get

$$\int \frac{1}{(1+x^2)\tan^{-1}x}\,dx = -\int \frac{1}{y}\,dy$$

$$\int \frac{\frac{1}{1+x^2}}{\tan^{-1}x}\cdot dx = -\int \frac{1}{y}\,dx$$

This is of the form $\int \frac{f'(x)}{f(x)}dx = \log[f(x)]$

$$\log(\tan^{-1}x) = -\log y + \log c$$

$$\log(\tan^{-1}x) + \log y = \log c$$

$$\log[(\tan^{-1}x)y] = \log c$$

$\therefore\ y(\tan^{-1}x) = C$ is the soln.

13) $\tan y \cdot \frac{dy}{dx} = \sin(x+y) + \sin(x-y)$

soln:- $\tan y\,dy = [\sin(x+y) + \sin(x-4)]\,dx$

$\tan y\,dy = [\sin x\cdot\cos y + \cancel{\cos x\,\sin y} + \sin x\cos y - \cancel{\cos x\,\sin y}]\,dx$

$\tan y\,dy = 2\sin x\cdot\cos y\,dx$

$\frac{\tan y\,dy}{\cos y} = 2\sin x\,dx$ $\quad$ $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

$(\tan y\cdot\sec y)\,dy = 2\sin x\,dx$

Integrating on both side, we get.

$$\int(\tan y\cdot\sec y)\,dy = 2\int\sin x\,dx$$

$$\sec y = -2\cos x + c$$

$2\cos x + \sec y = C$ is the soln.

14) $\frac{dy}{dx} = (1+x)(1+y^2)$ given $y=1$ when $x=0$.

soln :- $\frac{dy}{dx} = (1+x)(1+y^2)$

$\frac{dy}{1+y^2} = (1+x)\,dx$

Integrating on both side, we get

$\int \frac{1}{1+y^2}\,dy = \int dx + \int x\,dx$

$\tan^{-1} y = x + \frac{x^2}{2} + C$ is the soln $\longrightarrow$ ①

Given that $y=1$ when $x=0$.

① => $\tan^{-1} 1 = 0 + 0 + C$

$\tan^{-1} 1 = C$

$\frac{\pi}{4} = C$

② => ~~to~~ Substitute the value of C in eqn ①

we get

$\tan^{-1} y = x + \frac{x^2}{2} + \frac{\pi}{4}$

15) If $\frac{dy}{dx} = e^{x+y}$ and it is given that for $x=1$ & $y=1$ then find y when $x=1$.

soln :- Given $\frac{dy}{dx} = e^{x+y}$

$\frac{dy}{dx} = e^x \cdot e^y$

$\frac{dy}{e^y} = e^x\,dx$

$e^{-y}\,dy = e^x\,dx$

Integrating on both side, we get

$\int e^{-y}\,dy = \int e^x\,dx$

$$-e^{-y} = e^{x} + c \longrightarrow (1)$$

put $x=1$ and $y=1$ in (1), we get

$$-e^{-1} = e + c$$

$$c = -\frac{1}{e} - e = -\left[\frac{1}{e} + e\right] = -\left[\frac{1+e^2}{e}\right].$$

$$\therefore c = -\left[\frac{1+e^2}{e}\right]$$

Substitute the value of 'c' in (1), we get

$$-e^{-y} = e^{x} - \left[\frac{1+e^2}{e}\right] \longrightarrow (2)$$

Now when $x=1$, we have,

$$(2) \Rightarrow -e^{-y} = e - \left[\frac{1+e^2}{e}\right]$$

$$-e^{-y} = \frac{\cancel{e^2} - 1 - \cancel{e^2}}{e}$$

$$\cancel{-}e^{-y} = \cancel{-}\frac{1}{e}$$

$$e^{-y} = e^{-1}$$

$$\cancel{-}y = \cancel{-}1$$

$$\boxed{y = 1}.$$

HW

16) $\dfrac{dy}{dx} = \dfrac{y^2(x+1)}{x^2(y+1)}$

soln :- $\dfrac{dy}{dx} = \dfrac{y^2(x+1)}{x^2(y+1)}$

$$\left(\frac{y+1}{y^2}\right) dy = \left(\frac{x+1}{x^2}\right) dx$$

$$\left(\frac{\cancel{y}}{y^{\cancel{2}}} + \frac{1}{y^2}\right) dy = \left(\frac{\cancel{x}}{x^{\cancel{2}}} + \frac{1}{x^2}\right) dx$$

~~log~~ Integrating on both side, we get

$$\int \frac{1}{y}\,dy + \int \frac{1}{y^2}\,dy = \int \frac{1}{x}\,dx + \int \frac{1}{x^2}\,dx$$

$$\log y - \frac{1}{y} = \log x - \frac{1}{x} + c$$

$$\log y - \log x - \frac{1}{y} + \frac{1}{x} = c$$

$$\log\left[\frac{y}{x}\right] - \left[\frac{1}{y} - \frac{1}{x}\right] = c$$

HW

17) Solve $(x - y^2x)\,dx - (y - x^2y)\,dy = 0$

soln :- $x(1-y^2)\,dx = y(1-x^2)\,dy$

$$\frac{x}{1-x^2}\,dx = \frac{y}{1-y^2}\,dy$$

Integrating on both side, we get

$$\int \frac{x}{1-x^2}\,dx = \int \frac{y}{1-y^2}\,dy$$

multiple & divided by (-2) on both side, we get

$$-\frac{1}{2}\int \frac{-2x}{1-x^2}\,dx = -\frac{1}{2}\int \frac{-2y}{1-y^2}\,dy$$

$$-\frac{1}{2}\log(1-x^2) = -\frac{1}{2}\log(1-y^2) + \log c$$

$$-\log(\sqrt{1-x^2}) = -\log(\sqrt{1-y^2}) + \log c$$

$$\log(\sqrt{1-y^2}) - \log(\sqrt{1-x^2}) = \log c$$

$$\log\left[\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\right] = \log c$$

$$\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = c \quad \text{or} \quad \sqrt{1-y^2} = c(\sqrt{1-x^2})$$

This is the soln.

HW

18) Find the eqn of the curve represented by $(y - xy)dx + (x + xy)dy = 0$ and passing through the point $(1, 1)$.

soln :- Given that $(y - xy)dx + (x + xy)dy = 0$

$$y(1-x)\,dx = -x(1+y)\,dy$$

$$\frac{(1-x)}{x}dx = -\left(\frac{1+y}{y}\right)dy$$

Integrating on both side, we get

$$\int \frac{1}{x}dx - \int 1\,dx = -\int \frac{1}{y}dy - \int 1\,dy$$

$$\log x - x = -\log y - y + C$$

$$\log y + \log x - x + y = C$$

$$\log(xy) - x + y = C \longrightarrow ①$$

Given the points $(1,1)$ i.e., $(x,y) = (1,1)$

$\Rightarrow x = 1, y = 1$.

put $x = 1$, $y = 1$ in eqn ①, we get

$$\log(1 \cdot 1) - \cancel{1} + \cancel{1} = C$$

$$\boxed{C = 0}$$

Substitute $C = 0$ in eqn ①, we get

$$\log(xy) - x + y = 0$$

HW

19). $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}}$ given that $y=1$ when $x=0$.

soln :-

$$\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}} \qquad \left\{\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}\right\}$$

$$\frac{dy}{dx} = \frac{-\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\frac{1}{\sqrt{1-y^2}}dy = \frac{-1}{\sqrt{1-x^2}}.dx$$

rough the

Integrating on both side, we get

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = -\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\sin^{-1} y = -\sin^{-1} x + c$$

$$\sin^{-1} y + \sin^{-1} x = c \longrightarrow ①$$

given that $y = 1$, when $x = 0$, then ① implies.

$$① \Rightarrow \sin^{-1} 1 + \sin^{-1} 0 = c$$

$$\frac{\pi}{2} + 0 = c$$

$$\boxed{c = \frac{\pi}{2}}$$

Substitute the $c = \frac{\pi}{2}$ in eqn ①, we get.

$$\sin^{-1} y + \sin^{-1} x = \frac{\pi}{2}$$

HW

2) $(1-y)\,x\,\frac{dy}{dx} + (1+x)\,y = 0$. given $y = 1$ when $x = 1$.

soln:-

$$(1-y)\,x\,\frac{dy}{dx} = -(1+x)\,y$$

$$\frac{(1-y)}{y}\,dy = -\frac{(1+x)}{x}\,dx$$

$$\left(\frac{1}{y} - 1\right) dy = -\left(\frac{1}{x} + 1\right) dx$$

Integrating on both side, we get.

$$\int \frac{1}{y}\,dy - \int 1\,dy = -\int \frac{1}{x}\,dx - \int 1\,dx$$

$$\log y - y = -\log x - x + c$$

$$\log y + \log x - y + x = c$$

$$\log(xy) - y + x = c \longrightarrow ①$$

given $y = 1$, when $x = 1$, we get.

$$① \Rightarrow \log(\cancel{1})^0 - \cancel{1} + \cancel{1} = c$$

$$\boxed{c = 0}$$

$$\therefore ① \Rightarrow \log(xy) - y + x = 0$$

Equation Reducible to Variable Separable form:

* The D.E is of the form $\frac{dy}{dx} = f(ax+by+c)$ can be solved by reducing it to variable separable form using the substitution $ax+by+c = v$

* If the given D.E is of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ where $\frac{a}{a'} = \frac{b}{b'}$

In this case we take the substitution $ax+by=v$ to reduce the variable separable form.

Problems :-

1) Solve $\frac{dy}{dx} = (4x+y+1)^2$

soln :- Given $\frac{dy}{dx} = (4x+y+1)^2 \longrightarrow (1)$

put $4x+y+1 = v$

Diff w.r.t x on b.s

$$4 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 4$$

from (1) $\Rightarrow$ $\frac{dv}{dx} - 4 = v^2$

$$\frac{dv}{dx} = v^2 + 4$$

$$\frac{dv}{v^2+4} = dx$$

Integrating on both side we get

$$\int \frac{dv}{v^2+4} = \int dx$$

$$\left\{ \text{WKT} : \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c \right\}$$

$$\int \frac{1}{v^2+2^2}\, dv = \int dx$$

$$\frac{1}{2} \tan^{-1}\left(\frac{v}{2}\right) = x + C.$$

$$\frac{1}{2} \tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + C$$ is the soln

2) Solve $(x+y)^2 \frac{dy}{dx} = a^2$.

soln :- Given $(x+y)^2 \frac{dy}{dx} = a^2$

$$\frac{dy}{dx} = \frac{a^2}{(x+y)^2} \longrightarrow (1)$$

put $x + y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

from (1) $\Rightarrow$ $$\frac{dv}{dx} - 1 = \frac{a^2}{v^2}$$

$$\frac{dv}{dx} = \frac{a^2}{v^2} + 1$$

$$\frac{dv}{dx} = \frac{a^2+v^2}{v^2}$$

$$\frac{v^2}{a^2+v^2}\, dv = dx$$

Integrating on both side, we get

$$\int \frac{v^2}{a^2+v^2}\, dv = \int dv$$

add and substract a^2 on LHS of the eqn, we get.

$$\int \frac{(v^2+a^2)-a^2}{v^2+a^2}\,dv = \int dx$$

$$\int \frac{\cancel{v^2+a^2}}{\cancel{v^2+a^2}}\,dv - \int \frac{a^2}{v^2+a^2}\,dv = \int dx$$

$$v - a^{\cancel{2}}\frac{1}{\cancel{a}}\tan^{-1}\left(\frac{v}{a}\right) = x + C$$

$$v - a\tan^{-1}\left(\frac{v}{a}\right) = x + C$$

$$\cancel{x}+y - a\tan^{-1}\left(\frac{x+y}{a}\right) = \cancel{x} + C$$

$$y - a\tan^{-1}\left(\frac{x+y}{a}\right) = C \text{ is the soln}$$

HW

3) Solve $\frac{dy}{dx} = (3x+2y+1)^2$

Soln:- Given $\frac{dy}{dx} = (3x+2y+1)^2 \longrightarrow (1)$

put $3x+2y+1 = v$

Diff w.r.t x on both side.

$$3 + 2\frac{dy}{dx} = \frac{dv}{dx}$$

$$2\frac{dy}{dx} = \frac{dv}{dx} - 3$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{dv}{dx} - 3\right]$$

from (1) $\Rightarrow$ $$\frac{1}{2}\left[\frac{dv}{dx} - 3\right] = v^2$$

$$\frac{dv}{dx} - 3 = 2v^2$$

$$\frac{dv}{dx} = 2v^2 + 3$$

$$\frac{dv}{2v^2+3} = dx$$

Integrating on b.s. we get

we get.

$$\int \frac{dv}{2v^2+3} = \int dx$$

$$\frac{1}{2}\int \frac{1}{v^2+(\sqrt{3/2})^2} dv = \int dx$$

This is of the form $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$.

$$\frac{1}{2}\left[\frac{1}{\sqrt{3/2}} \cdot \tan^{-1}\frac{v}{\sqrt{3/2}}\right] = x + C$$

$$\frac{1}{2}\,\frac{\sqrt{2}}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{2}v}{\sqrt{3}}\right) = x + C$$

$$\frac{1}{\sqrt{2}\cdot\cancel{\sqrt{2}}}\cdot\frac{\cancel{\sqrt{2}}}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{2}v}{\sqrt{3}}\right) = x + C$$

$$\frac{1}{\sqrt{6}}\tan^{-1}\left[\frac{\sqrt{2}}{\sqrt{3}}(3x+2y+1)\right] = x + C$$ is the soln

HW

4) $(x+y)^2 \frac{dy}{dx} = a^2$

soln:- given $(x-y)^2 \frac{dy}{dx} = a^2$

$$\frac{dy}{dx} = \frac{a^2}{(x-y)^2}$$

put $x - y = v$

$$1 - \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{dv}{dx}$$

$$1 - \frac{dv}{dx} = \frac{a^2}{v^2}$$

$$\frac{dv}{dx} = 1 - \frac{a^2}{v^2}$$

$$\frac{dv}{dx} = \frac{v^2 - a^2}{v^2}$$

$$\frac{v^2}{v^2 - a^2}\,dv = dx$$

Integrating on both side, we get

$$\int \frac{v^2}{v^2-a^2}\, dv = \int dx$$

add & substract a^2 in LHS of the above eqn

$$\int \frac{v^2-a^2+a^2}{v^2-a^2}\, dv = \int dx$$

$$\int \frac{\cancel{v^2-a^2}}{\cancel{v^2-a^2}}\, dv + \int \frac{a^2}{v^2+a^2}\, dv = \int dx$$

$$v + \cancel{a^2}\,\frac{1}{2\cancel{a}} \log\left[\frac{v-a}{v+a}\right] = x + C$$

$$v + \frac{a}{2} \log\left[\frac{v-a}{v+a}\right] = x + C$$

$$\cancel{x} - y + \frac{a}{2} \log\left[\frac{x-y-a}{x-y+a}\right] = \cancel{x} + C$$

$$-y + \frac{a}{2} \log\left[\frac{x-y-a}{x-y+a}\right] = C$$ is a soln

* $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$
* $\int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a} \log\left[\frac{x-a}{x+a}\right] + c$
* $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a} \log\left[\frac{a+x}{a-x}\right] + c$

Integrating on both side, we get

$$\int \frac{v^2}{v^2-a^2}\,dv = \int dx$$

add & substract a^2 in LHS of the above eqn

$$\int \frac{v^2-a^2+a^2}{v^2-a^2}\,dv = \int dx$$

$$\int \frac{\cancel{v^2-a^2}}{\cancel{v^2-a^2}}\,dv + \int \frac{a^2}{v^2+a^2}\,dv = \int dx$$

$$v + \cancel{a^2}\,\frac{1}{2\cancel{a}} \log\left[\frac{v-a}{v+a}\right] = x + C$$

$$v + \frac{a}{2} \log\left[\frac{v-a}{v+a}\right] = x + C$$

$$\cancel{x} - y + \frac{a}{2} \log\left[\frac{x-y-a}{x-y+a}\right] = \cancel{x} + C$$

$$-y + \frac{a}{2} \log\left[\frac{x-y-a}{x-y+a}\right] = C$$ is a soln.

* $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
* $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left[\frac{x-a}{x+a}\right] + c$
* $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left[\frac{a+x}{a-x}\right] + c$

5m

5) Solve $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Soln:- Given $\frac{dy}{dx} = \sin(x+y) + \cos(x+y) \longrightarrow (1)$

put $x+y = v$.

Diff. w.r.t x.

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

(1) $\Rightarrow$ $\frac{dv}{dx} - 1 = \sin v + \cos v$

$$\frac{dv}{dx} = \sin v + \cos v + 1$$

$$\frac{dv}{1+\cos v+\sin v} = dx.$$

Integrating on both side we get

$$\int \frac{1}{1+\cos v+\sin v}\,dv = \int dx.$$

Formula

$1+\cos v = 2\cos^2(v/2)$

$\sin v = 2\sin(\frac{v}{2})\cos(\frac{v}{2})$

$$\int \frac{1}{2\cos^2(\frac{v}{2}) + 2\sin(v/2)\cos(v/2)}\,dv = \int dx$$

$$\int \frac{1}{2\cos^2(\frac{v}{2})\left[1+\frac{\sin(v/2)}{\cos(v/2)}\right]}\,dv = \int dx$$

$$\int \frac{\sec^2(v/2)}{2\left[1+\tan(v/2)\right]}\,dv = \int dx.$$

$$\log\left[1+\tan\left(\frac{v}{2}\right)\right] = x + C$$

$\log\left[1+\tan\left(\frac{x+y}{2}\right)\right] = x + C$ is the required soln

HW

Q) Solve $\frac{dy}{dx} = 1 + 6x e^{x+y}$

Soln:- Given $\frac{dy}{dx} = 1 + 6x e^{x-y} \longrightarrow (1)$

put $x - y = v$

Diff w.r.t. x

$$1 - \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{dv}{dx}$$

$$(1) \Rightarrow \cancel{1} - \frac{dv}{dx} = \cancel{1} + 6x e^{v}$$

$$\frac{dv}{dx} = -6x e^{v}$$

Separate the variable. we get

$$\frac{dv}{e^{v}} = -6x\,dx$$

$$e^{-v}\,dv = -6x\,dx$$

Integrating on both side, we get.

$$\int e^{-v}\,dv = -6\int x\,dx$$

$$\cancel{-}e^{-v} = \cancel{-}\overset{3}{\cancel{6}}\left(\frac{x^2}{\cancel{2}} + c\right)$$

$$e^{-v} = 3x^2 + 3c \qquad \text{put } c_1 = 3c$$

$$e^{-(x-y)} = 3x^2 + c_1$$

$$e^{y-x} = 3x^2 + c_1 \text{ is the soln}$$

7) Solve $\frac{dy}{dx} = \frac{6x - 2y - 7}{3x - y + 4}$

Soln:- Given $\frac{dy}{dx} = \frac{6x-2y-7}{3x-y+4}$

$$\frac{dy}{dx} = \frac{2(3x - y) - 7}{3x - y + 4} \longrightarrow (1)$$

Here $\frac{a}{a'} = \frac{b}{b'}$

$\Rightarrow \frac{3}{3} = \frac{1}{1}$

$\Rightarrow \frac{1}{1} = \frac{1}{1}$

put $3x - y = v$

$$3 - \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = 3 - \frac{dv}{dx}$$

from (1) $\Rightarrow$ $3 - \frac{dv}{dx} = \frac{2v - 7}{v + 4}$

$$\frac{dv}{dx} = 3 - \left[\frac{2v-7}{v+4}\right]$$

$$\frac{dv}{dx} = \frac{3v + 12 - 2v + 7}{v + 4}$$

$$\frac{dv}{dx} = \frac{v + 19}{v + 4}$$

Separate the variable, we get

$$\frac{v + 4}{v + 19}\, dv = dx$$

Integrating on both side we get

$$\int \frac{v + 4}{v + 19}\, dv = \int dx$$

$$\int \frac{v + 19 - 15}{v + 19}\, dv = \int dx.$$

$$\int \cancel{\frac{v + 19}{v + 19}}\, dv - \int \frac{15}{v + 19}\, dv = \int dx.$$

$$v - 15 \log [v + 19] = x + C$$

$$3x - y - 15 \log [\cancel{3x-y}+ 19] - x \mp C$$

$2x - y - 15 \log (3x - y + 19) = C$ is the soln.

8) $$\frac{dy}{dx} = \frac{1 - 3x - 3y}{2(x + y)}$$

soln:- given $$\frac{dy}{dx} = \frac{1 - 3x - 3y}{2(x + y)}$$

$$\frac{dy}{dx} = \frac{1 - 3(x + y)}{2(x + y)} \longrightarrow (1)$$

put $x + y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$① \Rightarrow \quad \frac{dv}{dx} - 1 = \frac{1-3v}{2v}$$

$$\frac{dv}{dx} = \frac{1-3v}{2v} + 1$$

$$\frac{dv}{dx} = \frac{1-3v+2v}{2v}$$

$$\frac{dv}{dx} = \frac{1-v}{2v}$$

$$\frac{2v}{1-v}\, dv = dx$$

Integrating on both side, we get

$$\int \frac{2v}{1-v}\, dv = \int dx$$

$$\int \frac{2v-2+2}{1-v}\, dv = \int dx$$

$$\int \frac{-2(1-v)+2}{(1-v)}\, dv = \int dx$$

$$-2\int \frac{\cancel{1-v}}{\cancel{1-v}}\, dv + 2\int \frac{1}{1-v}\, dv = \int dx$$

$$-2v + 2\log(1-v) = x + C$$

$$-2(x+y) + 2\log(1-x-y) = x + C$$

$$-2x - 2y + 2\log(1-x-y) - x = C$$

$-3x - 2y + 2\log(1-x-y) = C$ is the soln.

?) Solve $\frac{dy}{dx} = \frac{4x+6y+5}{2x+3y+4}$

Soln:- Given $\frac{dy}{dx} = \frac{4x+6y+5}{2x+3y+4}$

$\frac{a}{a'} = \frac{b}{b'} \Rightarrow \frac{4}{2} = \frac{6}{3}$

$\Rightarrow 2 = 2$

$$\frac{dy}{dx} = \frac{2(2x+3y)+5}{2x+3y+4} \longrightarrow (1)$$

put $2x + 3y = v$

$$2 + 3\frac{dy}{dx} = \frac{dv}{dx}$$

$$3\frac{dy}{dx} = \frac{dv}{dx} - 2$$

$$\frac{dy}{dx} = \frac{1}{3}\left[\frac{dv}{dx} - 2\right]$$

$$(1) \Rightarrow \frac{1}{3}\left[\frac{dv}{dx} - 2\right] = \frac{2v+5}{v+4}$$

$$\frac{dv}{dx} - 2 = \frac{6v+15}{v+4}$$

$$\frac{dv}{dx} = \frac{6v+15}{v+4} + 2$$

$$\frac{dv}{dx} = \frac{6v+15+2v+8}{v+4}$$

$$\frac{dv}{dx} = \frac{8v+23}{v+4}$$

Separate the variable.

$$\frac{v+4}{8v+23}\,dv = dx$$

Integrating on b.s.

$$\int \frac{v+4}{8v+23}\,dv = \int dx$$

multiply ~~and divided by~~ by 8 on b.s

$$\int \frac{8v+32}{8v+23}\,dv = \int 8dx$$

$$\int \frac{8v+23+9}{8v+23}\,dv = \int 8dx$$

$$\int \frac{8v+23}{8v+23}\,dv + \int \frac{9}{8v+23}\,dv = 8\int dx$$

$$v + 9\int \frac{1}{8\left[v+\frac{23}{8}\right]}\,dv = 8x + C$$

$$v + \frac{9}{8}\log\left[v+\frac{23}{8}\right] = 8x + C$$

$$2x+3y+\frac{9}{8}\log\left[2x+3y+\frac{23}{8}\right] = 8x+C$$ is the required soln:

$$\int \frac{8v+23}{8v+23}\,dv + \int \frac{9}{8v+23}\,dv = 8\int dx$$

$$v + 9\int \frac{1}{8\left[v+\frac{23}{8}\right]}\,dv = 8x + C$$

$$v + \frac{9}{8}\log\left[v+\frac{23}{8}\right] = 8x + C$$

$2x+3y+\frac{9}{8}\log\left[2x+3y+\frac{23}{8}\right] = 8x+C$ is the required soln:

Method 2.

Homogeneous Differential Equation

Homogenous means the power of variables in each term of the expression is same.

Eg :- ① x^2-y^2 ② $x^2+3xy+y^2$.
③ $x^3+y^3+2x^2y$.

A Homogenous differential Equation is of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$

where $f(x,y)$ and $g(x,y)$ are homogenous func of the same degree in x and y.

To Solve a homogeneous Differential Equation:-

1) put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$
2) Separate the variable v and x and integrate
3) Replace v by $\frac{y}{x}$ to get the required soln.

Problems:-

1) Solve $(x^2+y^2)\,dx + 2xy\,dy = 0$

Soln:- Given $(x^2+y^2)\,dx + 2xy\,dy = 0$

$$\frac{dy}{dx} = -\frac{(x^2+y^2)}{2xy} \longrightarrow (1)$$

put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1)

$$(1) \Rightarrow v + x\frac{dv}{dx} = -\frac{(x^2+v^2x^2)}{2x(vx)}$$

$$= -\frac{\cancel{x^2}(1+v^2)}{2\cancel{x^2}\,v}$$

$$v + x\frac{dv}{dx} = -\frac{(1+v^2)}{2v}$$

$$x\frac{dv}{dx} = -\frac{(1+v^2)}{2v} - v$$

$$= \frac{-1-v^2-2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{-1-3v^2}{2v} = \frac{-(1+3v^2)}{2v}$$

Separate the variable, we get

$$\frac{2v}{1+3v^2}\,dv = -\frac{1}{x}\,dx$$

Multiple 3 on both side

$$\frac{6v}{1+3v^2}\,dv = -\frac{3}{x}\,dx$$

Integrating on both side, we get

$$\int\frac{6v}{1+3v^2}\,dv = -3\int\frac{1}{x}\,dx$$

$$\log[1+3v^2] = -3\log x + \log c$$

$$\log[1+3v^2] + \log x^3 = \log c$$

$$\log\left[(1+3v^2)\cdot x^3\right] = \log c$$

$$x^3(1+3v^2) = c.$$

But $\quad y = vx \Rightarrow v = \frac{y}{x}$

$$x^3\left[1+3\frac{y^2}{x^2}\right] = c$$

$$\cancel{x^3}\left[\frac{x^2+3y^2}{\cancel{x^2}}\right] = c$$

$$x\left[x^2+3y^2\right] = c$$

$x^3 + 3xy^2 = c$ is the required soln.

2) Solve $(x-y)\,dy - (2x-y)\,dx = 0$

Soln:- The given eqn can be written as $\frac{dy}{dx} = \frac{2x-y}{x-y} \rightarrow (1)$

put $\quad y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1)

$$(1) \Rightarrow v + x\frac{dv}{dx} = \frac{2x - vx}{x - vx}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{x}(2-v)}{\cancel{x}(1-v)}$$

$$x\frac{dv}{dx} = \frac{2-v}{1-v} - v$$

$$x\frac{dv}{dx} = \frac{2-v-v+v^2}{1-v}$$

$$x\frac{dv}{dx} = \frac{v^2-2v+2}{1-v}$$

Separate the variable we get

$$\frac{1-v}{v^2-2v+2}\,dv = \frac{1}{x}\,dx$$

$$\frac{-(v-1)}{v^2-2v+2}\,dv = \frac{1}{x}\,dx$$

$$\int \frac{v-1}{v^2-2v+2}\,dv = -\int \frac{1}{x}\,dx$$

multiply by 2 on both side.

$$\int \frac{2v-2}{v^2-2v+2}\,dv = -2\int \frac{1}{x}\,dx$$

$$\log[v^2-2v+2] = -2\log x + \log c$$

$$\log[v^2-2v+2] + \log x^2 = \log c$$

$$\log[x^2(v^2-2v+2)] = \log c$$

$$x^2(v^2-2v+2) = c$$

But $y = vx \Rightarrow v = \frac{y}{x}$

$$x^2\left[\frac{y^2}{x^2} - 2\frac{y}{x} + 2\right] = c$$

$$\cancel{x^2}\left[\frac{y^2-2xy+2x^2}{\cancel{x^2}}\right] = c$$

$y^2 - 2xy + 2x^2 = c$ is the required soln.

3) Solve $\left[x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right]y - \left[y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right]x\frac{dy}{dx} = 0$.

Soln: Given $\left[x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right]y - \left[y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right]x\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{y\left[x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right]}{x\left[y\sin\left(\frac{y}{x}\right) - x\cos\left(y/x\right)\right]} \longrightarrow (1)$$

put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1)

$$(1) \Rightarrow v + x\frac{dv}{dx} = \frac{vx[x\cos v + y\sin v]}{x[vx\sin v - x\cos v]}$$

$$v + x\frac{dv}{dx} = \frac{v\cancel{x^2}[\cos v + v\sin v]}{\cancel{x^2}[v\sin v - \cos v]}$$

$$x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$= \frac{v\cos v + v^2\cancel{\sin v} - v^2\cancel{\sin v} + v\cos v}{v\sin v - \cos v}$$

$$x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}.$$

Separate the variable, we get

$$\frac{v\sin v - \cos v}{v\cos v}dv = 2\left(\frac{1}{x}\right)dx$$

Integrating on both side,

$$\int\frac{v\sin v - \cos v}{v\cos v}dv = 2\int\frac{1}{x}dx$$

$$-\int\frac{\cos v - v\sin v}{v\cos v}dv = 2\int\frac{1}{x}dx$$

$$-\log[v\cos v] = 2\log x + \log c.$$

$$\log[v\cos v] + \log x^2 + \log c = 0$$

$$\log[(v\cos v)x^2 \cdot c] = \log 1$$

$$(v\cos v)x^2 \cdot c = 1$$

But $y = vx \Rightarrow v = \frac{y}{x}$

$$\frac{y}{\cancel{x}}\cos\left(\frac{y}{x}\right) \cdot x^{\cancel{2}} \cdot c = 1$$

$xy\cos(y/x) = \frac{1}{c}$ is the required soln.

42) $\left[x \sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)\right]dx + \left[x\cos(y/x)\right]dy = 0$

soln:-

$$\frac{dy}{dx} = \frac{-\left[x \sin(y/x) - y\cos(y/x)\right]}{x\cos(y/x)} \longrightarrow (1)$$

put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1).

$$v + x\frac{dv}{dx} = \frac{-\left[x\sin v - vx\cos v\right]}{x\cos v}.$$

$$v + x\frac{dv}{dx} = \frac{-\cancel{x}\left[\sin v - v\cos v\right]}{\cancel{x}\cos v}$$

$$x\frac{dv}{dx} = \frac{-\sin v + v\cos v}{\cos v} - v$$

$$= \frac{-\sin v + \cancel{v\cos v} - \cancel{v\cos v}}{\cos v}.$$

$$x\frac{dv}{dx} = -\frac{\sin v}{\cos v}$$

Separate the variable, we get.

$$\frac{\cos v}{\sin v}\,dv = -\frac{1}{x}\,dx$$

Integrating on b.s, we get

$$\int \cot v\, dv = -\int \frac{1}{x}\,dx$$

$$\log \sin v = -\log x + \log c$$

$$\log \sin v + \log x = \log c$$

$$\log\left[x\sin v\right] = \log c$$

$$x\sin v = c$$

But $y = vx \Rightarrow v = y/x$

$$x\sin(y/x) = c$$

$x\sin\left(\frac{y}{x}\right) = c$ is the required soln.

5) $x\frac{dy}{dx} = y[\log y - \log x + 1]$

Soln :- $\frac{dy}{dx} = \frac{y[\log y - \log x + 1]}{x} \longrightarrow ①$

put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

① => $v + x\frac{dv}{dx} = \frac{v\cancel{x}[\log(vx) - \log x + 1]}{\cancel{x}}$

$v + x\frac{dv}{dx} = v[\log v + \cancel{\log x} - \cancel{\log x} + 1]$

$v + x\frac{dv}{dx} = v\log v + v$

$x\frac{dv}{dx} = v\log v \cancel{+v} \cancel{-v}$

Separate the variable, we get

$\int \frac{1}{v\log v} dv = \frac{1}{x} dx$

Integrating on both side, we get

$\int \frac{1/v}{\log v} dv = \int \frac{1}{x} dx$

$\log[\log v] = \log x + \log c$

$\log[\log v] = \log(xc)$

$\log v = xc$

put $v = y/x$

$\log(y/x) = xc$ is the required soln

6) Solve $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

soln: $y^2 + x^2\frac{dy}{dx} - xy\frac{dy}{dx} = 0$

$y^2 + (x^2 - xy)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-y^2}{(x^2 - xy)} \longrightarrow ①$

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

① $\Rightarrow$ $$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2 - x^2v}$$

$$= \frac{-v^2x^2}{\cancel{x^2}(1-v)}$$

$$v + x\frac{dv}{dx} = \frac{-v^2}{1-v}$$

$$x\frac{dv}{dx} = \frac{-v^2}{1-v} - v$$

$$= \frac{-\cancel{v^2} - v + \cancel{v^2}}{1-v}$$

$$x\frac{dv}{dx} = \frac{-v}{1-v}$$

Separate the variable, we get

$$\frac{1-v}{v}\,dv = -\frac{1}{x}\,dx$$

Integrating on both side.

$$\int\frac{1}{v}\,dv - \int dv = -\int\frac{1}{x}\,dx$$

$$\log v - v = -\log x + c$$

$$\log v + \log x = v + C$$

$$\log(vx) = v + C$$

put $v = y/x$

$$\log\left[\frac{y}{\cancel{x}}\cdot\cancel{x}\right] = \frac{y}{x} + C$$

$\log(y) = \frac{y}{x} + C$ is the required soln.

7) Solve $(y^2 + 2xy)\,dx + (2x^2 + 3xy)\,dy = 0$

Soln :-

$$\frac{dy}{dx} = \frac{-(y^2 + 2xy)}{2x^2 + 3xy} \longrightarrow ①$$

put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

① $\Rightarrow$ $$v + x\frac{dv}{dx} = \frac{-(v^2x^2 + 2vx^2)}{2x^2 + 3vx^2}$$

$$v + x\frac{dv}{dx} = \frac{-x^2(v^2+2v)}{x^2[2+3v]}$$

$$x\frac{dv}{dx} = \frac{-v^2-2v}{2+3v} - v$$

$$x\frac{dv}{dx} = \frac{-v^2-2v-2v-3v^2}{2+3v}$$

$$= \frac{-4v^2-4v}{2+3v}$$

$$x\frac{dv}{dx} = \frac{-4(v^2+v)}{2+3v}$$

Separate the variable, we get

$$\frac{3v+2}{v^2+v}\,dv = -\frac{4}{x}\,dx$$

Integrating on both side, we get

$$\int \frac{3v+2}{v(v+1)}\,dv = -4\int \frac{1}{x}\,dx \longrightarrow (2)$$

Consider $\frac{3v+2}{v(v+1)} = \frac{A}{v} + \frac{B}{v+1} \longrightarrow (*)$

$$3v+2 = A(v+1) + Bv$$

put $v=0 \Rightarrow 2 = A+0$

$$\boxed{A=2}$$

put $v=-1 \Rightarrow -3+2 = 0+B(-1) \Rightarrow \cancel{+}1 = \cancel{+}B$

$$\boxed{B=1}$$

Substitute the value of A & B in (*), we get

$$\frac{3v+2}{v(v+1)} = \frac{2}{v} + \frac{1}{v+1}$$

Substitute the above value in (2).

$$\int\left(\frac{2}{v} + \frac{1}{v+1}\right)dv = -4\int \frac{1}{x}\,dx$$

$2\log v + \log(v+1) = -4\log x + \log c$

$\log v^2 + \log(v+1) + \log x^4 = \log c$

$\log[v^2(v+1) \cdot x^4] = \log c$

$v^2(v+1)x^4 = c$

put $v = \frac{y}{x}$

$\frac{y^2}{\cancel{x^2}}\left[\frac{y}{x}+1\right]x^{\cancel{4}} = c$

$y^2\left[\frac{y+x}{\cancel{x}}\right]x^{\cancel{2}} = c$

$xy^2[y+x] = c$ is the required soln.

HW

4) Solve $x\dfrac{dy}{dx}+\dfrac{y^2}{x}=y$

Solution: Given $x\dfrac{dy}{dx}+\dfrac{y^2}{x}=y \Rightarrow x\dfrac{dy}{dx}=y-\dfrac{y^2}{x}$

$$\frac{dy}{dx}=\frac{xy-y^2}{x^2}$$

$y=vx$, then $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{x(vx)-v^2x^2}{x^2}$$

$$v + x\frac{dv}{dx} = \frac{x^2(v - v^2)}{x^2} \Rightarrow x\frac{dv}{dx} = v - v^2 - v$$

$$x\frac{dv}{dx} = -v^2$$

Separate the variable we get

$$\frac{dv}{v^2} = -\frac{dx}{x}$$

Integrate we get

$$\int\frac{dv}{v^2} = -\int\frac{dx}{x} \Rightarrow -\frac{1}{v} = -\log x + c$$

$\log x - \frac{x}{y} = c$ is the required solution.

8)**Solve** $\left(x\tan\frac{y}{x}-y\sec^2\frac{y}{x}\right)dx+x\sec^2\left(\frac{y}{x}\right)dy=0$

Solution: The given equation can be written as

$$\frac{dy}{dx}=-\left[\frac{x\tan\frac{y}{x}-y\sec^2\frac{y}{x}}{x\sec^2\frac{y}{x}}\right],\ \text{Put}\ \ y=vx\Rightarrow v=\frac{y}{x}\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=-\left[\frac{x\tan v-vx\sec^2 v}{x\sec^2 v}\right]$$

$$\Rightarrow v+x\frac{dv}{dx}=-\frac{\tan v}{\sec^2 v}+v\Rightarrow x\frac{dv}{dx}=-\frac{\tan v}{\sec^2 v}$$

Separate the variable $\frac{\sec^2 v}{\tan v}dv=-\frac{dx}{x}$, Integrate we get

$$\int\frac{\sec^2 v}{\tan v}dv+\int\frac{dx}{x}=\log c\Rightarrow\log(\tan v)+\log x=\log c$$

$$\log[x\ \tan v]=\log c\Rightarrow x\tan\left(\frac{y}{x}\right)=c.$$

Equations Reducible to Homogenous Form :-

The D.E of the form $\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1} \longrightarrow (1)$

can be reduced to Homogenous forms as follows.

Case 1 :- When $\frac{a}{a_1} \neq \frac{b}{b_1}$,

then put $x = X+h$ and $y = Y+k$

where h, k are constants.

So that $dx = dX$, and $dy = dY$.

$$(1) \Rightarrow \frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a_1(X+h)+b_1(Y+k)+c_1}$$

$$= \frac{aX+ah+bY+bk+c}{a_1X+a_1h+b_1Y+b_1k+c_1}$$

$$= \frac{aX+bY+(ah+bk+c)}{a_1X+b_1Y+(a_1h+b_1k+c_1)} \longrightarrow (2)$$

Choose h, k. so that eqn (2) may become homogeneous.

$\therefore$ Put $ah+bk+c=0$ and $a_1h+b_1k+c_1=0$

So that $\frac{dY}{dX} = \frac{aX+bY}{a_1X+b_1Y}$.

which is homogeneous in X and Y and can be Solved by putting $Y = vX$

h and k values are given by solving the eqns $ah+bk+c=0$ and $a_1h+b_1k+c_1=0$.

Problems:-

1) Solve $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$.

soln:- $\frac{dy}{dx} = \frac{y+x-2}{y-x-4} \quad \left[\frac{a}{a_1} \neq \frac{b}{b_1}\right] \longrightarrow (1)$.

put $x = X+h$, $y = Y+k$ and $\frac{dy}{dx} = \frac{dY}{dX}$

$(2) \Rightarrow \frac{dY}{dX} = \frac{Y+k+X+h-2}{Y+k-X-h-4}$

$= \frac{Y+X+(k+h-2)}{Y-X+(k-h-4)} \longrightarrow (2)$

choose h, k such that $k+h-2=0$. and
$k-h-4=0$.

$\Rightarrow k=3$. and $h=-1$

$(2) \Rightarrow \frac{dY}{dX} = \frac{Y+X}{Y-X} \longrightarrow (3)$

put $Y = vX \Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$

$(3) \Rightarrow v + X\frac{dv}{dX} = \frac{vX+X}{vX-X}$

$v + X\frac{dv}{dX} = \frac{\cancel{X}(v+1)}{\cancel{X}(v-1)}$

$X\frac{dv}{dX} = \frac{v+1}{v-1} - v$

$X\frac{dv}{dX} = \frac{v+1-v^2+v}{v-1}$

$X\frac{dv}{dX} = \frac{-v^2+2v+1}{v-1}$

$$x \frac{dv}{dx} = \frac{-(v^2-2v-1)}{v-1}$$

Separate the variable

$$\frac{v-1}{v^2-2v-1} dv = -\frac{1}{x} dx$$

Integrating on both side, we get

$$\int \frac{v-1}{v^2-2v-1} dv = -\int \frac{1}{x} dx$$

multiply by 2 on both side, we get

$$\int \frac{2v-2}{v^2-2v-1} dv = -2\int \frac{1}{x} dx$$

$$\log(v^2-2v-1) = -2 \log x + \log c$$

$$\log(v^2-2v-1) + \log x^2 = \log c$$

$$\log[x^2(v^2-2v-1)] = \log c$$

$$x^2(v^2-2v-1) = c$$

put $v = \frac{y}{x}$

$$x^2\left[\frac{y^2}{x^2} - 2\frac{y}{x} - 1\right] = c$$

$$\cancel{x^2}\left[\frac{y^2-2xy-x^2}{\cancel{x^2}}\right] = c$$

$$y^2 - 2xy - x^2 = c \longrightarrow (4)$$

But $Y = y-k$ and $X = x-h$

$Y = y-3$ $\quad X = x+1 \quad (\because k=3, h=-1)$

Substitute the value of Y and X in (4)

$$(y-3)^2 - 2(x+1)(y-3) - (x+1)^2 = c$$

$$y^2+9-6y-2[xy-3x+y-3]-(x^2+1+2x) = c$$

$y^2 + 9 - 6y - 2xy + 6x - 2y + 6 - x^2 + 1 - 2x = C$

$y^2 - x^2 - 2xy + 4x - 8y + 14 = C$ is the required soln.

2) Solve $\frac{dy}{dx} = \frac{2x+y+6}{-x+y-3}$ when $x=0$ and $y=0$

soln :- $\frac{dy}{dx} = \frac{2x+y+6}{-x+y-3}$ $\left[\frac{a}{a_1} \neq \frac{b}{b_1}\right] \longrightarrow ①$

put $x = X+h$, $y = Y+k$ and $\frac{dy}{dx} = \frac{dY}{dX}$

① => $\frac{dY}{dX} = \frac{2(X+h)+Y+k+6}{-X-h+Y+k-3}$

$= \frac{2X+Y+(2h+k+6)}{-X+Y+(k-h-3)} \longrightarrow ②$

Choose h, k ∋ $\underbrace{2h+k+6=0}_{\to ⓐ}$ and $\underbrace{k-h-3=0}_{\to ⓑ}$

Solve the eqn ⓐ and ⓑ, we get $\boxed{h=-3}$, $\boxed{k=0}$

② => $\frac{dY}{dX} = \frac{2X+Y}{-X+Y} \longrightarrow ③$

put $Y = vX$ then $\frac{dY}{dX} = v + X\frac{dv}{dX}$

③ => $v + X\frac{dv}{dX} = \frac{2X+vX}{-X+vX}$

$= \frac{\cancel{X}[2+v]}{\cancel{X}[-1+v]}$

$X\frac{dv}{dX} = \frac{v+2}{v-1} - v$

$= \frac{v+2-v^2+v}{v-1}$

$X\frac{dv}{dX} = \frac{-v^2+2v+2}{v-1} = \frac{-(v^2-2v-2)}{v-1}$

Separate the variable we get

$$\frac{v-1}{v^2-2v-2}\,dv = -\frac{1}{X}\,dX$$

Integrating on both side, we get

$$\int \frac{v-1}{v^2-2v-2}\,dv = -\int \frac{1}{X}\,dX.$$

multiply by 2 on both side, we get

$$\int \frac{2v-2}{v^2-2v-2}\,dv = -2\int \frac{1}{X}\,dX$$

$$\log(v^2-2v-2) = -2\log X + \log c$$

$$\log(v^2-2v-2) + \log X^2 = \log c$$

$$\log[X^2[v^2-2v-2]] = \log c$$

$$X^2[v^2-2v-2] = c$$

put $v = \frac{Y}{X}$

$$X^2\left[\frac{Y^2}{X^2} - 2\frac{Y}{X} - 2\right] = c$$

$$\cancel{X^2}\left[\frac{Y^2-2XY-2X^2}{\cancel{X^2}}\right] = c$$

$$Y^2 - 2XY - 2X^2 = c \longrightarrow (4)$$

But $Y = y-k$, $X = x-h$,

$Y = y-0$ $\quad X = x+3$

Substitute the value of Y and X in (4)

$$(4) \Rightarrow (y^2) - 2(x+3)y - 2(x+3)^2 = c$$

$$y^2 - 2xy - 6y - 2x^2 - 18 - 12x = c$$

$$\underline{\underline{y^2 - 2x^2 - 2xy - 6y - 12x - 18 = c}} \longrightarrow (5)$$

Given $x=0$ and $y=0$.

(5) $\Rightarrow$ $-18 = C$ $\Rightarrow$ $\boxed{C=-18}$

Substitute the value of C in (5).

$\therefore$ (5) $\Rightarrow$ $y^2 - 2x^2 - 2xy - 6y - 12x \cancel{-18} = \cancel{-18}$

$y^2 - 2x^2 - 2xy - 6y - 12x = 0$ is the required soln.

3) Solve $(2x+3y-5)\frac{dy}{dx} = (3x+2y-5)$

Soln:- $\frac{dy}{dx} = \frac{3x+2y-5}{2x+3y-5} \rightarrow ① \left[\frac{a}{a_1} \neq \frac{b}{b_1}\right]$

put $x = X+h$, $y = Y+k$ and $\frac{dy}{dx} = \frac{dY}{dX}$

$$① \Rightarrow \frac{dY}{dX} = \frac{3(X+h)+2(Y+k)-5}{2(X+h)+3(Y+k)-5}$$

$$= \frac{3X+2Y+(3h+2k-5)}{2X+3Y+(2h+3k-5)} \longrightarrow ②$$

choose h, k, $\ni$ $3h+2k-5=0 \hookrightarrow ⓐ$, $2h+3k-5=0 \hookrightarrow ⓑ$

Solve eqn ⓐ and ⓑ,

$$\begin{aligned} 3h+2k-5&=0 \quad \times 2 \\ 2h+3k-5&=0 \quad \times 3 \\ \hline \cancel{6h}+4k-10&=0 \\ \cancel{6h}+9k-15&=0 \\ (-)\quad (-)\quad\ & (+) \\ \hline -5k+5&=0 \\ \cancel{+}5k&=\cancel{+}5 \\ \boxed{k=1} \end{aligned}$$

$$ⓐ \Rightarrow 3h+2(1)-5=0$$

$$3h-3=0$$

$$\boxed{h=1}$$

$$② \Rightarrow \frac{dY}{dX} = \frac{3X+2Y}{2X+3Y} \longrightarrow ③$$

put $Y = vX$ then $\frac{dY}{dX} = v + X\frac{dv}{dX}$ in ③

$$③ \Rightarrow v + X\frac{dv}{dX} = \frac{3X+2vX}{2X+3vX}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{x}(3+2v)}{\cancel{x}(2+3v)}$$

$$x\frac{dv}{dx} = \frac{3+2v}{2+3v} - v$$

$$x\frac{dv}{dx} = \frac{3+\cancel{2v}-\cancel{2v}-3v^2}{2+3v}$$

$$x\frac{dv}{dx} = \frac{3-3v^2}{2+3v} = \frac{-3(v^2-1)}{2+3v}$$

Separate the variable, we get.

$$\frac{3v+2}{v^2-1}\,dv = \frac{-3}{x}\,dx$$

Integrating on b.s, we get

$$\int\frac{3v+2}{v^2-1}.\,dv = -3\int\frac{1}{x}\,dx$$

$$\int\frac{3v}{v^2-1}\,dv + 2\int\frac{1}{v^2-1}.\,dv = -3\int\frac{1}{x}\,dx$$

Multiple and divided by 2.

$$\frac{3}{2}\int\frac{2v}{v^2-1}.\,dv + 2\int\frac{1}{v^2-1}.\,dv = -3\int\frac{1}{x}\,dx$$

$$\frac{3}{2}\log(v^2-1) + \cancel{2}\cdot\frac{1}{\cancel{2}}\log\left[\frac{v-1}{v+1}\right] = -3\log x + \log c.$$

[Formula : $\int\frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left[\frac{x-a}{x+a}\right]$]

$$\log(v^2-1)^{3/2} + \log\left[\frac{v-1}{v+1}\right] = -\log x^3 + \log c$$

$$\log\left[(v^2-1)^{3/2}\left(\frac{v-1}{v+1}\right)\right] = \log\left[\frac{c}{x^3}\right]$$

$$(v^2-1)^{3/2}\left[\frac{v-1}{v+1}\right] = \frac{c}{x^3} \longrightarrow (4)$$

put $v = \frac{Y}{X}$ in ④

$$④ \Rightarrow \left[\frac{Y^2}{X^2} - 1\right]^{3/2} \left[\frac{\frac{Y}{X} - 1}{\frac{Y}{X} + 1}\right] = \frac{C}{X^3}$$

$$\left[\frac{Y^2 - X^2}{X^2}\right]^{3/2} \left[\frac{Y - X}{Y + X}\right] = \frac{C}{X^3}$$

$$\frac{[Y^2 - X^2]^{3/2}}{[X^{\cancel{2}}]^{3/\cancel{2}}} \left[\frac{Y - X}{Y + X}\right] = \frac{C}{X^3}$$

$$\frac{(Y+X)^{3/2} \cdot (Y-X)^{3/2}}{\cancel{X^3}} \cdot \frac{(Y-X)}{(Y+X)} = \frac{C}{\cancel{X^3}}$$

$$(Y+X)^{3/2} \cdot (Y+X)^{-1} \cdot (Y-X)^{3/2} \cdot (Y-X) = C$$

$$(Y+X)^{3/2 - 1} \cdot (Y-X)^{3/2 + 1} = C$$

$$(Y+X)^{1/2} (Y-X)^{5/2} = C \longrightarrow ⑤$$

put $Y = y - k$, $X = x - h$.

$Y = y - 1$, $X = x - 1$ in ⑤

$$(y - 1 + x - 1)^{1/2} (y - \cancel{1} - x + \cancel{1})^{5/2} = C$$

$$(x + y - 2)^{1/2} (y - x)^{5/2} = C$$ is the required soln.

4) $$\frac{dy}{dx} = \frac{x + y + 1}{x - y} \longrightarrow ①$$

Soln :- put $x = X + h$, $y = Y + k$ and $\frac{dy}{dx} = \frac{dY}{dX}$ in ①

$$① \Rightarrow \frac{dY}{dX} = \frac{X + h + Y + k + 1}{X + h - Y - k}$$

$$= \frac{X + Y + (h + k + 1)}{X - Y + (h - k + 0)}$$

Choose, h, k $\ni$ $h+k+1=0$ (a) and $h-k=0$ (b)

Solving above two eqn we get the value of h, k.

$$\begin{array}{r} h+\cancel{k}+1=0 \\ h-\cancel{k}+0=0 \\ \hline 2h+1=0 \end{array}$$

$$\boxed{h=-1/2}$$

(a) $\Rightarrow -1/2+k+1=0$

$k = 1/2 - 1$

$$\boxed{k=-1/2}$$

$$\frac{dY}{dX} = \frac{X+Y}{X-Y} \longrightarrow (2)$$

put $Y = vX$ then $\frac{dY}{dX} = v + X\frac{dv}{dX}$ in (2).

$$v + X\frac{dv}{dX} = \frac{X+vX}{X-vX}$$

$$v + X\frac{dv}{dX} = \frac{\cancel{X}(1+v)}{\cancel{X}(1-v)}$$

$$X\frac{dv}{dX} = \frac{1+v}{1-v} - v$$

$$X\frac{dv}{dX} = \frac{1+\cancel{v}-\cancel{v}+v^2}{1-v}$$

$$X\frac{dv}{dX} = \frac{v^2+1}{1-v} \neq \frac{\cancel{v^2+1}}{\cancel{-(v-1)}} =$$

Separate the variable, we get

$$\frac{1-v}{v^2+1}\,dv = \frac{1}{X}\,dX$$

Integrating on b.s, we get

$$\int\frac{1-v}{v^2+1}\,dv = \int\frac{1}{X}\,dX$$

$$\int\frac{1}{v^2+1}\,dv - \int\frac{v}{v^2+1}\,dv = \int\frac{1}{X}\,dX$$

$\downarrow$

x^{ly} and $\div$ by 2

$$\int \frac{1}{v^2+1}\,dv - \frac{1}{2}\int \frac{2v}{v^2+1}\,dv = \int \frac{1}{x}\,dx$$

$$\tan^{-1}(v) - \frac{1}{2}\log(v^2+1) = \log x + \log c$$

$$\tan^{-1}(v) - \frac{1}{2}\log(v^2+1) - \log x = c$$

$$\tan^{-1}(v) - \log x\sqrt{v^2+1} = c.$$

put $v = \frac{Y}{X}$. in above soln.

$$\tan^{-1}\left[\frac{Y}{X}\right] - \log\left[X\sqrt{Y^2/X^2+1}\right] = c$$

$$\tan^{-1}\left[\frac{Y}{X}\right] - \log\left[\cancel{X}\frac{\sqrt{Y^2+X^2}}{\cancel{X}}\right] = c \longrightarrow (3)$$

Substitute $Y = y-k$, $X = x-h$

$Y = y+1/2$, $X = x+1/2$ in (3), we get

$$\tan^{-1}\left[\frac{y+1/2}{[x+1/2]}\right] - \log\left[\sqrt{(y+1/2)^2+(x+1/2)^2}\right] = c$$

$$\tan^{-1}\left[\frac{\frac{2y+1}{\cancel{2}}}{\left[\frac{2x+1}{\cancel{2}}\right]}\right] - \log\left[\sqrt{y^2+1/4+y+x^2+1/4+x}\right] = c$$

$$\tan^{-1}\left[\frac{2y+1}{(2x+1)}\right] - \log\left[y^2+x^2+y+x+1/2\right]^{1/2} = c$$

$$\tan^{-1}\left[\frac{2y+1}{2x+1}\right] - \frac{1}{2}\log(x^2+y^2+x+y+1/2) = c$$

This is the required soln of given D.E.

Case 2 :- When $\frac{a}{a_1} = \frac{b}{b_1}$ i.e., $ab_1 - a_1b = 0$

the case (i) fails as h, k becomes infinite or indeterminate

Now $\frac{a}{a_1} = \frac{b}{b_1} = \frac{1}{m}$ (say)

$\therefore \frac{a}{a_1} = \frac{1}{m}, \quad \frac{b}{b_1} = \frac{1}{m}$

$a_1 = am, \quad b_1 = bm$

$\therefore$ The D.E of the form $\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1} \longrightarrow (1)$

Substitute the value of a_1, b_1 in (1), we get

$$\frac{dy}{dx} = \frac{ax+by+c}{amx+bmy+c_1}$$

$$\frac{dy}{dx} = \frac{ax+by+c}{m(ax+by)+c_1} \longrightarrow (2)$$

Now put $ax+by = v$ so that $a + b\frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b}\left[\frac{dv}{dx} - a\right]$$

$$(2) \Rightarrow \frac{1}{b}\left[\frac{dv}{dx} - a\right] = \frac{v+c}{mv+c_1}$$

$$\frac{dv}{dx} - a = \frac{bv+bc}{mv+c_1}$$

$$\frac{dv}{dx} = \frac{bv+bc}{mv+c_1} + a$$

$$\frac{dv}{dx} = \frac{bv+bc+amv+ac_1}{mv+c_1}$$

$$\frac{dv}{dx} = \frac{(am+b)v + (ac_1 + bc)}{mv + c_1}$$

So that we can separate the variable and integrating, we get the soln.

In the solution, put $v = ax + by$.

1) Solve $\frac{dy}{dx} = \frac{x+2y+1}{2x+4y+3} \rightarrow ①$

Soln:- Here $\frac{a}{a_1} = \frac{b}{b_1} \Rightarrow \frac{1}{2} = \frac{2}{4} \Rightarrow \frac{1}{2} = \frac{1}{2}$

~~put~~. $\frac{dy}{dx} = \frac{x+2y+1}{2(x+2y)+3} \longrightarrow ①$

put $x + 2y = v \Rightarrow 1 + 2\frac{dy}{dx} = \frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{dv}{dx} - 1\right]$$

$① \Rightarrow \frac{1}{2}\left[\frac{dv}{dx} - 1\right] = \frac{v+1}{2v+3}$

$$\frac{dv}{dx} - 1 = \frac{2v+2}{2v+3}$$

$$\frac{dv}{dx} = \frac{2v+2}{2v+3} + 1$$

$$= \frac{2v+2+2v+3}{2v+3}$$

$$\frac{dv}{dx} = \frac{4v+5}{2v+3}$$

$$\frac{2v+3}{4v+5}\, dv = dx$$

Integrating on both side, we get

$$\int \frac{2v+3}{4v+5} \cdot dv = \int dx$$

Multiple by 2 on b.s

$$\int \frac{4v+6}{4v+5}\, dv = 2\int dx$$

$$\int \frac{4v+5+1}{4v+5} \cdot dv = 2\int dx$$

$$\int \frac{\cancel{4v+5}}{\cancel{4v+5}} \cdot dv + \int \frac{1}{4v+5} \cdot dv = 2\int dx$$

$$\int dv + \frac{1}{4}\int \frac{1}{v+5/4} \cdot dv = 2\int dx$$

$$v + \frac{1}{4}\log\left(v+5/4\right) = 2x + C \longrightarrow (2)$$

put $v = x + 2y$ in (2)

$$x+2y + \frac{1}{4}\log\left(x+2y+5/4\right) = 2x + C$$

$2y - x + \frac{1}{4}\log\left(x+2y+5/4\right) = C$ is the required soln.

Linear Differential Equation :-

A D.E is said to be linear. if

(i) All derivatives and dependent variables are of degree 1.

(ii) There is no product term of dependent variable and its derivatives or no product of two derivatives (i.e., , yy', $y'y''$, $y'y''$, ----)

Thus the standard form of a L.D.E of the first order commonly known as Leibnitz's linear Equation.

The Std form of a L.D.E of first order, first degree is $\boxed{\frac{dy}{dx} + Py = Q}$

where P and Q are functions of x. or consts

Note :-

* A Non-Linear D.E is a D.E which is not linear

Ex :- 1) $y'' + \sqrt{y} = 0 \longrightarrow$ N.L.D.E [$\because$ D.V. power is not equal to 1]

2) $y'' + y = \cos x \longrightarrow$ L.D.E.

3) $xy' + y = 2 \longrightarrow$ L.D.E.

4) $\frac{dy}{dx} = \frac{x}{y} \longrightarrow$ NLDE [$\because$ product of DV and its derivatives present]

5) $\frac{dy}{dx} = \frac{y}{x} \longrightarrow$ LDE.

6) $y'' + \sin y = x \longrightarrow$ NLDE [$\because$ transcendental func of D.V is present]

* In Linear D.E, there is no transcendental func of D.V or its derivatives present.

But transcendental func of I V doesn't make difference.

for example :- ① $y'' + \sin y = 1$ is N.L D.E but

$y'' + y = \sin x$ is L.D.E.

② $y'' + e^y = x$ is N L D.E but

$y'' + y = e^x$ is L.D.E

[trignometric func, exponential func and logarithmic func is transcendental func]

Working Rule :-

1) The standard form of L.D.E is $\frac{dy}{dx} + Py = Q$.

2) Find the Integral factor [I.F]

i.e., $IF = e^{\int P dx}$

3) The soln is of the form $y(IF) = \int Q(IF)\, dx + C$.

Note :-

Sometimes a D.E becomes linear if we take 'y' as the I.V and x as D.V. In this case the eqn can be written as $\frac{dx}{dy} + Px = Q$.

where P and Q are func of y or constants.

In this case IF is $e^{\int P dy}$ and soln can be written as $x(IF) = \int Q(IF)\, dy + C$

Problems :-

1) Solve $(x+1)\frac{dy}{dx} - y = e^{2x}(x+1)^2$

soln :- $\div$ by $(x+1)$, we get

$$\frac{dy}{dx} - \left(\frac{1}{x+1}\right) y = e^{2x}(x+1)$$

Here $P = \frac{-1}{x+1}$, $Q = e^{2x}(x+1)$

$$I.F = e^{\int P dx} = e^{-\int \frac{1}{x+1} dx} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

$\therefore$ The Soln is $y\,(IF) = \int Q\,(IF)\,dx + C$

$$y\left[\frac{1}{x+1}\right] = \int e^{2x}\,\cancel{(x+1)}\cdot\left[\frac{1}{\cancel{x+1}}\right] dx + C$$

$$y\left[\frac{1}{x+1}\right] = \int e^{2x}\,dx + C$$

$$\frac{y}{x+1} = \frac{e^{2x}}{2} + C$$

$\therefore$ $y = \left(\frac{e^{2x}}{2} + c\right)(x+1)$ is the soln

2) Solve $x\frac{dy}{dx} + 2y - x^2 \log x = 0$

soln:

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

$\div$ throughout by x Here $P = 2/x$

$$\frac{dy}{dx} + \frac{2}{x} = x \log x \qquad Q = x \log x$$

$$IF = e^{\int P dx} = e^{\int 2/x \, dx} = e^{2\log x} = e^{\log x^2} = \underline{\underline{x^2}}$$

The soln is

$$y(IF) = \int Q(IF)\, dx + C$$

$$y(x^2) = \int x \cdot \log x \,(x^2)\, dx + C$$

$$yx^2 = \int \underset{II}{x^3} \underset{I}{\log x}\, dx + C$$

$$yx^2 = \log x \left(\frac{x^4}{4}\right) - \frac{x^5}{20} \cdot \frac{1}{x} + C$$

$$yx^2 = \log x \left(\frac{x^4}{4}\right) - \frac{x^4}{20} + C$$ is the required soln

3) Solve $\sec x \frac{dy}{dx} = y + \sin x$

soln :- $\div \sec x$ on both side, we get

$$\frac{dy}{dx} = \frac{1}{\sec x} y + \frac{\sin x}{\sec x}$$

$$\frac{dy}{dx} - \cos x \, y = \sin x . \cos x$$

$P = -\cos x$ $\quad Q = \sin x \cos x$

$$IF = e^{\int P dx} = e^{-\int \cos x \, dx} = e^{-\sin x}$$

$\therefore$ Hence the soln is

$$y(IF) = \int Q(IF)\, dx + C$$

$$y e^{-\sin x} = \int \sin x . \cos x . e^{-\sin x} . dx + C$$

For RHS, put $\sin x = t$

$\cos x \, dx = dt$

$$y e^{-\sin x} = \int t e^{-t} . dt + C$$

$$y e^{-\sin x} = t(-e^{-t}) - e^{-t} + C$$

$$y e^{-\sin x} = -e^{-t}(t+1) + C$$

$$y e^{-\sin x} = -e^{-\sin x}(\sin x + 1) + C$$

$$[y + (\sin x + 1)] e^{-\sin x} = C$$

$$y + \sin x + 1 = \frac{C}{e^{-\sin x}}$$

$$y + \sin x + 1 = C e^{\sin x}$$ is the required soln.

6) Solve $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$ Given $x=0$ and $y=0$

Soln:- $\div$ by $(1-x^2)$ on b.s

$$\frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x\sqrt{1-x^2}}{(1-x^2)}$$

$$\frac{dy}{dx} + \frac{2x}{1-x^2}y = x(1-x^2)^{-1/2}$$

Here $P = \frac{2x}{1-x^2}$ $Q = \frac{x}{\sqrt{1-x^2}}$

$$IF = e^{\int P dx} = e^{-\int \frac{-2x}{1-x^2}dx} = e^{-\log(1-x^2)} = (1-x^2)^{-1} = \frac{1}{1-x^2}$$

$\therefore$ The soln is of the form $y(IF) = \int Q(IF)dx + C$

$$y\left(\frac{1}{1-x^2}\right) = \int \frac{x}{\sqrt{1-x^2}} \cdot \left(\frac{1}{1-x^2}\right) dx + C$$

$$= \int \frac{x}{(1-x^2)^{3/2}} \cdot dx + C$$

put $1-x^2 = t \Rightarrow -2x \cdot dx = dt$

$\Rightarrow x\,dx = -dt/2$

$$y\left(\frac{1}{1-x^2}\right) = \int \frac{1}{(t)^{3/2}}\left(\frac{-dt}{2}\right) + C$$

$$= \frac{-1}{2}\int (t)^{-3/2} \cdot dt + C$$

$$= -\frac{1}{2}\left(\frac{t^{-3/2+1}}{-3/2+1}\right) + C$$

$$= -\frac{1}{\cancel{2}}\left(\frac{t^{-1/2}}{-1/\cancel{2}}\right) + C$$

$$\frac{y}{1-x^2} = \frac{1}{\sqrt{t}} + C$$

But $t = 1-x^2$

$$\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + C \longrightarrow (1)$$

given $x=0$, $y=0$

$$0 = 1 + C$$

$$\Rightarrow \boxed{C = -1}$$

Substitute the value of c in (1), we get

$$\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} - 1$$ is the required soln