Mathematical Induction

Mathematical induction, is a technique for proving results or establishing statements for natural numbers.

Definition

Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.

The technique involves two steps to prove a statement, as stated below −

Step 1(Base step) − It proves that a statement is true for the initial value.

Step 2(Inductive step) − It proves that if the statement is true for the n^{th} iteration (or number *n*), then it is also true for *(n+1)th* iteration (or number *n+1*).

How to Do It

Step 1 − Consider an initial value for which the statement is true. It is to be shown that the statement is true for n = initial value.

Step 2 − Assume the statement is true for any value of *n* = *k*. Then prove the statement is true for *n* = *k+1*. We actually break *n* = *k+1* into two parts, one part is *n* = *k* (which is already proved) and try to prove the other part.

Problem 1

$3^n - 1$ is a multiple of 2 for n = 1, 2, ...

Solution

Step 1 − For $n = 1, 3^1 - 1 = 3 - 1 = 2$ which is a multiple of 2

Step 2 − Let us assume $3^n - 1$ is true for $n = k$, Hence, $3^k - 1$ is true (It is an assumption)

We have to prove that $3^{k+1} - 1$ is also a multiple of 2

$$3^{k+1} - 1 = 3 \times 3^k - 1 = (2 \times 3^k) + (3^k - 1)$$

The first part $(2 \times 3k)$ is certain to be a multiple of 2 and the second part $(3k - 1)$ is also true as our previous assumption.

Hence, $3^{k+1}-1$ is a multiple of 2.

So, it is proved that 3^n-1 is a multiple of 2.

Problem 2

$1+3+5+\ldots+(2n-1)=n^2$ for $n=1,2,\ldots$

Solution

Step 1 − For $n=1, 1=1^2$, Hence, step 1 is satisfied.

Step 2 − Let us assume the statement is true for $n=k$.

Hence, $1+3+5+\cdots+(2k-1)=k^2$ is true (It is an assumption)

We have to prove that $1+3+5+\ldots+(2(k+1)-1)=(k+1)^2$ also holds

$$1+3+5+\cdots+(2(k+1)-1)$$

$$=1+3+5+\cdots+(2k+2-1)$$

$$=1+3+5+\cdots+(2k+1)$$

$$=1+3+5+\cdots+(2k-1)+(2k+1)$$

$$=k^2+(2k+1)$$

$$=(k+1)^2$$

So, $1+3+5+\cdots+(2(k+1)-1)=(k+1)^2$ hold which satisfies the step 2.

Hence, $1+3+5+\cdots+(2n-1)=n^2$ is proved.

Problem 3

Prove that $(ab)^n = a^n b^n$ is true for every natural number n

Solution

Step 1 − For $n = 1, (ab)^1 = a^1 b^1 = ab$, Hence, step 1 is satisfied.

Step 2 − Let us assume the statement is true for $n = k$, Hence, $(ab)^k = a^k b^k$ is true (It is an assumption).

We have to prove that $(ab)^{k+1} = a^{k+1} b^{k+1}$ also hold

Given, $(ab)^k = a^k b^k$

Or, $(ab)^k (ab) = (a^k b^k)(ab)$ [Multiplying both side by 'ab']

Or, $(ab)^{k+1} = (aa^k)(bb^k)$

Or, $(ab)^{k+1} = (a^{k+1} b^{k+1})$

Hence, step 2 is proved.

So, $(ab)^n = a^n b^n$ is true for every natural number n.

Strong Induction

Strong Induction is another form of mathematical induction. Through this induction technique, we can prove that a propositional function, $P(n)$ is true for all positive integers, n , using the following steps −

- **Step 1(Base step)** − It proves that the initial proposition $P(1)$ true.
- **Step 2(Inductive step)** − It proves that the conditional statement $[P(1) \land P(2) \land P(3) \land \cdots \land P(k)] \rightarrow P(k+1)$ is true for positive integers k .

Example: Adding up Odd Numbers

$$1 + 3 + 5 + \dots + (2n-1) = n^2$$

1. Show it is true for **n=1**

$$1 = 1^2 \text{ is True}$$

2. Assume it is true for **n=k**

$$1 + 3 + 5 + \dots + (2k-1) = k^2 \text{ is True}$$

(An assumption!)

Now, prove it is true for "k+1"

$$1 + 3 + 5 + \dots + (2k-1) + (2(k+1)-1) = (k+1)^2 \quad ?$$

We know that $\mathbf{1 + 3 + 5 + \dots + (2k-1) = k^2}$ (the assumption above), so we can do a replacement for all but the last term:

$$\mathbf{k^2} + (2(k+1)-1) = (k+1)^2$$

Now expand all terms:

$$k^2 + 2k + 2 - 1 = k^2 + 2k+1$$

And simplify:

$$k^2 + 2k + 1 = k^2 + 2k + 1$$

They are the same! So it is true.

So:

$$1 + 3 + 5 + \dots + (2(k+1)-1) = (k+1)^2 \text{ is True}$$

DONE!

Fundamental Counting Principle Definition.

The Fundamental Counting Principle (also called the counting rule) is a way to figure out the number of outcomes in a probability problem. Basically, you multiply the events together to get the total number of outcomes. The formula is:

If you have an event "a" and another event "b" then all the different outcomes for the events is

a * b.

Fundamental counting principle: Sample problem #1

A fast-food restaurant has a meal special: $5 for a drink, sandwich, side item and dessert. The choices are:

- Sandwich: Grilled chicken, All Beef Patty, Veg burger and Fish Filet.
- Side: Regular fries, Cheese Fries, Potato Wedges.
- Dessert: Chocolate Chip Cookie or Apple Pie.
- Drink: Fanta, Dr. Pepper, Coke, Diet Coke and Sprite.

Q. How many meal combos are possible?
A. There are 4 stages:

1. Choose a sandwich.
2. Choose a side.
3. Choose a dessert.
4. Choose a drink.

There are 4 different types of sandwich, 3 different types of side, 2 different types of desserts and five different types of drink.

The number of meal combos possible is 4 * 3 * 2 * 5 = 120.

Fundamental counting principle: Sample problem #2.

Q. You take a survey with five “yes” or “no” answers. How many different ways could you complete the survey?

A. There are 5 stages: Question 1, question 2, question 3, question 4, and question 5.
There are 2 choices for each question (Yes or No).
So the total number of possible ways to answer is:
2 * 2 * 2 * 2 * 2 = 32.

Fundamental counting principle: Sample problem #3.

Q: A company puts a code on each different product they sell. The code is made up of 3 numbers and 2 letters. How many different codes are possible?
A. There are 5 stages (number 1, number 2, number 3, letter 1 and letter 2).
There are 10 possible numbers: 0 – 9.
There are 26 possible letters: A – Z.
So we have:
10 * 10 * 10 * 26 * 26 = 676000 possible codes.

Fundamental Counting Principle Problems:

Question 1: You toss three dimes. How many possible outcomes are there?
- 2 * 2 * 2 = 8.

Question 2: Your school offers two English classes, three math classes and three history classes. You want to take one of each class. How many different ways are there to organize your schedule?
- 2 * 3 * 3 = 18.

Question 3: A wedding caterer gives you three choices for the main course, six starter choices and five options for dessert. How many different meals (made up of starter, dinner and dessert) are there?
- 3 * 6 * 5 = 90.

Question 4: You take a multiple choice test made up of 10 questions. Each question has 4 possible answers. How many different ways are there to answer the test (assuming you don’t leave a question blank)?
- 4 * 4 * 4 * 4 * 4 * 4 * 4 * 4 * 4 * 4 = 1048576.

Question 5: An online company is offering a date night special: pick one movie from four choices, one restaurant from six choices and either flowers, chocolates or wine. How many possible date night options are there?

- 4 * 6 * 3 = 72.

Permutation

In mathematics, permutation is the act of arranging the members of a set into a sequence or order, or, if the set is already ordered, rearranging (reordering) its elements—a process called permuting.

A **permutation** is an arrangement, or listing, of objects in which the order is important.

Example 1:

If five digits 1, 2, 3, 4, 5 are being given and a three digit code has to be made from it if the repetition of digits is allowed then how many such codes can be formed.

Solution:

As repetition is allowed, we have five options for each digit of the code. Hence, the required number of ways code can be formed is, $5 \times 5 \times 5 = 125$.

Example 2:

If three alphabets are to be chosen from A, B, C, D and E such that repetition is not allowed then in how many ways it can be done?

Solution:

The number of ways three alphabets can be chosen from five will be,

$${}^{3}_{5}P = \frac{5!}{(5-3)!} = \frac{5\times4\times3\times2\times1}{2\times1} = 60.$$

Hence, there are 60 possible ways.

Problem 1:

In how many ways can the letters of the word APPLE can be rearranged?

Solution:

Total number of alphabets in APPLE = 5.

Number of repeated alphabets = 2

Number of ways APPLE can be rearranged = $\frac{5!}{2!}$ = 60.

The word APPLE can be rearranged in 60 ways.

Problem 2:

10 students have appeared in a test in which the top three will get a prize. How many possible ways are there to get the prize winners?

Solution:

We need to choose and arrange 3 persons out of 10. Hence, the number of possible ways will be

${}^{3}_{10}\mathrm{P}$ = $\frac{10!}{(10-3)!}$ = $10 \times 9 \times 8$ = 720.

Problem 3:

Ellie want to change her password which is ELLIE9 but with same letters and number. In how many ways she can do that?

Solution:

Total number of letters = 6.

Repeated letters = 2 Ls and 2 Es.

Number of times ELLIE9 can be rearranged = $\frac{6!}{2!2!}$ = $6 \times 5 \times 3 \times 2 \times 1$ = 180.

But the password need to be changed. So, the number of ways new password can be made = $180 - 1 = 179$.

Problem 4:

In how many ways the word HOLIDAY can be rearranged such that the letter I will always come to the left of letter L?

Solution:

Number of letters in HOLIDAY = 7 and there is no repetition of letters. Hence, the number of ways all letters can be arranged is 7!.

The number of ways the letters are arranged such that I will come left of L will be, $\frac{7!}{2}$ as in half of the arrangements L will be right of I and in other half it will be on left of I.

Problem 5:

There are 6 people who will sit in a row but out of them Ronnie will always be left of Annie and Rachel will always be right of Annie. In how many ways such arrangement can be done?

Solution:

The total number of ways of 6 people being seated in a row will be 6!.

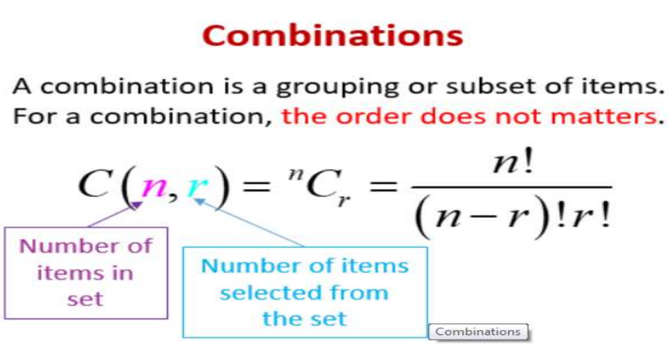
Now, with the given constraint the total number of ways will be $\frac{6!}{3!} = 6 \times 5 \times 4 = 120$.

It implies that out of 6 people arrangement of arrangement of 3 people is predefined.

Combination

A combination is a collection of the objects where the order doesn't matter. A combination is a mathematical technique that determines the number of possible arrangements in a collection of items where the order of the selection does not matter. In combinations, you can select the items in any order.

Combinations can be confused with permutations. Nevertheless, in permutations, the order of the selected items is essential. For example, the arrangements *ab* and *ba* are equal in combinations (considered as one arrangement) while in permutations, the arrangements are different.



Formula for Combination

Mathematically, the formula for determining the number of possible arrangements by selecting only a few objects from a set with no repetition is expressed in the following way:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Where:

- **n** – the total number of elements in a set
- **k** – the number of selected objects (the order of the objects is not important)
- **!** – factorial

Problems

Example : How many lines can you draw using 3 non collinear (not in a single line) points A, B and C on a plane?

Solution:

You need two points to draw a line. The order is not important. Line AB is the same as line BA. The problem is to select 2 points out of 3 to draw different lines. If we proceed as we did with permutations, we get the following pairs of points to draw lines.
AB , AC
BA , BC
CA , CB
There is a problem: line AB is the same as line BA, same for lines AC and CA and BC and CB.
The lines are: AB, BC and AC ; 3 lines only.
So in fact we can draw 3 lines and not 6 and that's because in this problem the order of the points A, B and C is not important.
This is a combination problem: combining 2 items out of 3 and is written as follows:

$${}_{n}C_{r} = n! / [\,(n - r)!\, r!\,]$$

The number of combinations is equal to the number of permutations divided by r! to eliminates those counted more than once because the order is not important.

Example : Calculate
${}_{3}C_{2}$
${}_{5}C_{5}$

Solution:

${}_{3}C_{2} = 3! / [\,(3 - 2)!2!\,] = 6 / [1 \times 2] = 3$
(problem of points and lines solved above in example 6)

${}_{5}C_{5} = 5! / [(5 - 5)!5!\,] = 5! / [0!5!] = 5! / [1 \times 5!] = 1$
(there is only one way to select (without order) 5 items from 5 items and to select all of them once!)

Example : We need to form a 5 a side team in a class of 12 students. How many different teams can be formed?

Solution:

There is nothing that indicates that the order in which the team members are selected is important and therefore it is a combination problem. Hence the number of teams is given by **$_{12}C_5 = 12! / [(12 - 5)!5!] = 792$**

Example:

In how many ways can a coach choose three swimmers from among five swimmers?

Solution:

There are 5 swimmers to be taken 3 at a time.

Using the formula:

The coach can choose the swimmers in 10 ways.

Example:

Six friends want to play enough games of chess to be sure every one plays everyone else. How many games will they have to play?

Solution:

There are 6 players to be taken 2 at a time.

Using the formula:

They will need to play 15 games.

Problems

1. How many 4 digit numbers can we make using the digits 3, 6, 7 and 8 without repetitions?
2. How many 3 digit numbers can we make using the digits 2, 3, 4, 5, and 6 without repetitions?
3. How many 6 letter words can we make using the letters in the word LIBERTY without repetitions?
4. In how many ways can you arrange 5 different books on a shelf?
5. In how many ways can you select a committee of 3 students out of 10 students?
6. How many triangles can you make using 6 non collinear points on a plane?
7. A committee including 3 boys and 4 girls is to be formed from a group of 10 boys and 12 girls. How many different committee can be formed from the group?
8. In a certain country, the car number plate is formed by 4 digits from the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 followed by 3 letters from the alphabet. How many number plates can be formed if neither the digits nor the letters are repeated?

Solutions to the above problems

1. $4! = 24$
2. $_5P_3 = 60$
3. $_7P_6 = 5040$
4. $5! = 120$
5. $_{10}C_3 = 120$
6. $_6C_3 = 20$
7. $_{10}C_3 \times {}_{12}c_4 = 59,400$
8. $_9P_4 \times {}_{26}P_3 = 47,174,400$

Permutations and Combinations

Number of permutations (order matters) of n things taken r at a time:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Number of combinations (order does not matter) of n things taken r at a time:

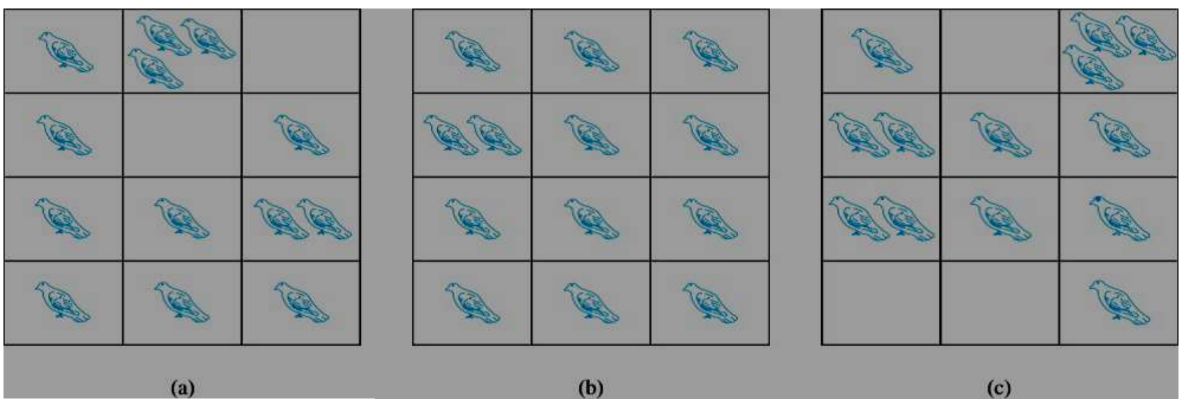
$$C(n,r) = \frac{n!}{(n-r)!r!}$$

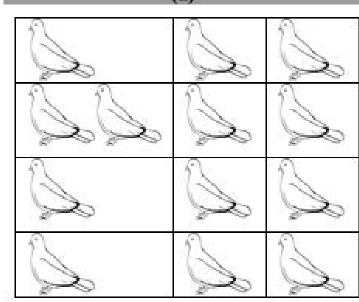
Number of different permutations of n objects where there are n_1 repeated items, n_2 repeated items, ... n_k repeated items

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

The Pigeonhole Principle

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it. To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most 19 pigeons, one per hole, could be accommodated. This illustrates a general principle called the pigeonhole principle, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it.





Theorem –

I) If “A” is the average number of pigeons per hole, where A is not an integer then

- At least one pigeon hole contains **ceil[A]** (smallest integer greater than or equal to A) pigeons
- Remaining pigeon holes contains at most **floor[A]** (largest integer less than or equal to A) pigeons

Or

II) We can say as, if n + 1 objects are put into n boxes, then at least one box contains two or more objects.

The abstract formulation of the principle: Let X and Y be finite sets and let $f : A \rightarrow B$ be a function.

- If X has more elements than Y, then f is not one-to-one.
- If X and Y have the same number of elements and f is onto, then f is one-to-one.
- If X and Y have the same number of elements and f is one-to-one, then f is onto.

- **Example – 1:** If (Kn+1) pigeons are kept in n pigeon holes where K is a positive integer, what is the average no. of pigeons per pigeon hole?

  Solution: average number of pigeons per hole = (Kn+1)/n
  = K + 1/n
  Therefore at least a pigeonholes contains (K+1) pigeons i.e., ceil[K +1/n] and remaining contain at most K i.e., floor[k+1/n] pigeons.
  i.e., the minimum number of pigeons required to ensure that at least one pigeon hole contains (K+1) pigeons is (Kn+1).

- **Example – 2:** A bag contains 10 red marbles, 10 white marbles, and 10 blue marbles. What is the minimum no. of marbles you have to choose randomly from the bag to ensure that we get 4 marbles of same color?

  Solution: Apply pigeonhole principle.
  No. of colors (pigeonholes) n = 3
  No. of marbles (pigeons) K+1 = 4
  Therefore the minimum no. of marbles required = Kn+1
  By simplifying we get Kn+1 = 10.
  Verification: ceil[Average] is [Kn+1/n] = 4
  [Kn+1/3] = 4
  Kn+1 = 10
  i.e., 3 red + 3 white + 3 blue + 1(red or white or blue) = 10

Pigeonhole principle strong form –

Theorem: Let $q_1, q_2, \ldots, q_n$ be positive integers. If $q_1 + q_2 + \ldots + q_n - n + 1$ objects are put into n boxes, then either the 1st box contains at least q_1 objects, or the 2nd box contains at least q_2 objects, . . ., the nth box contains at least q_n objects.

- **Example – 1:** In a computer science department, a student club can be formed with either 10 members from first year or 8 members from second year or 6 from third year or 4 from final year. What is the minimum no. of students we have to choose randomly from department to ensure that a student club is formed?

  Solution: we can directly apply from the above formula where,
  $q_1 = 10, q_2 = 8, q_3 = 6, q_4 = 4$ and n=4
  Therefore the minimum number of students required to ensure department club to be formed is
  10 + 8 + 6 + 4 – 4 + 1 = 25

- **Example – 2:** A box contains 6 red, 8 green, 10 blue, 12 yellow and 15 white balls. What is the minimum no. of balls we have to choose randomly from the box to ensure that we get 9 balls of same color?

  Solution: Here in this we cannot blindly apply pigeon principle. First we will see what happens if we apply above formula directly.

  From the above formula we have get answer 47 because 6 + 8 + 10 + 12 + 15 - 5 + 1 = 47

  But it is not correct. In order to get the correct answer we need to include only blue, yellow and white balls because red and green balls are less than 9. But we are picking randomly so we include after we apply pigeon principle.

  i.e., 9 blue + 9 yellow + 9 white – 3 + 1 = 25

  Since we are picking randomly so we can get all the red and green balls before the above 25 balls. Therefore we add 6 red + 8 green + 25 = 39

  We can conclude that in order to pick 9 balls of same color randomly, one has to pick 39 balls from a box.

Theorem 1.6.1 *(Pigeonhole Principle) Suppose that n+1 (or more) objects are put into n boxes. Then some box contains at least two objects.*

Proof.
Suppose each box contains at most one object. Then the total number of objects is at most 1+1+⋯+1=n1+1+⋯+1=n, a contradiction.

This seemingly simple fact can be used in surprising ways. The key typically is to put objects into boxes according to some rule, so that when two objects end up in the same box it is because they have some desired relationship.

Extended Pigeonhole Principle:

It states that if n pigeons are assigned to m pigeonholes (The number of pigeons is very large than the number of pigeonholes), then one of the pigeonholes must contain at least [(n-1)/m]+1 pigeons.

Proof: we can prove this by the method of contradiction. Assume that each pigeonhole does not contain more than [(n-1)/m] pigeons. Then, there will be at most

m[(n-1)/m]≤m(n-1)/m=n-1 pigeons in all. This is in contradiction to our assumptions. Hence, for given m pigeonholes, one of these must contain at least [(n-1)/m]+1 pigeons.

Problem:

solution : By extended pigeon hole principle at least $\left[\left|\frac{n-1}{m}\right|\right] + 1$ pigeons will occupy one pigeon hole.

Hence n =50, m=7

then 7 < 50

$$\begin{aligned}\left[\left|\frac{50-1}{7}\right|\right] + 1 &= 7 + 1 \\ &= 8\text{bicycles will be of same colour}\end{aligned}$$